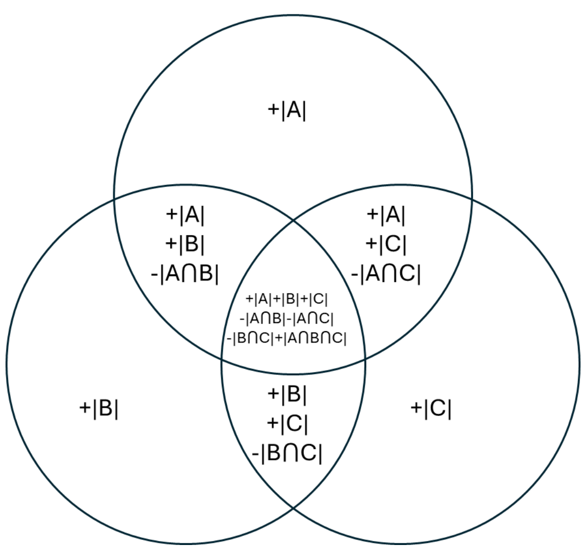
Worksheet 20: April 10 (Solutions)

Principles to Remember

- **Inclusion-Exclusion Principle:** When given a finite union of finite sets, this is how we find its size.
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - $|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$
  - $|A \cup B \cup C \cup D| = (|A|+|B|+|C|+|D|) - (|A \cap B|+|A \cap C|+|A \cap D|+|B \cap C|+|B \cap D|+|C \cap D|)+(|A \cap B \cap C|+|A \cap B \cap D|+|A \cap C \cap D|+|B \cap C \cap D|) - |A \cap B \cap C \cap D|$
  - $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|$
- **Simple recurrence relations:** Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and let $\beta = \dfrac{1-\sqrt{5}}{2} = \dfrac{1}{\alpha}$. If $\{a_n\}$ is a sequence, then $a_{n+2} = a_{n+1} + a_n$ if and only if there exist some $c, d \in \mathbb{R}$ such that $a_n = c\alpha^n + d\beta^n$.

Exercises

1. Draw Venn diagrams to illustrate the Inclusion-Exclusion Principle for unions of two and three sets.
   Solution: Behold the following crude diagram, made in PowerPoint because I don't actually know how to use image editing software.



2. Prove the Inclusion-Exclusion Principle. (*Hint:* Consider how many times an element belonging to exactly r of the A_i's is counted in each sum.)
Solution: Consider some x which belongs to exactly r of the A_i's. Then x is counted $\binom{r}{1}$ times in $\sum_i |A_i|$, counted $\binom{r}{2}$ times in $\sum_{i<j} |A_i \cap A_j|$, counted $\binom{r}{3}$ times in $\sum_{i<j<k} |A_i \cap A_j \cap A_k|$, and so forth. The net number of times x is counted in the inclusion-exclusion formula is therefore

$$\begin{aligned}
&\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \cdots + (-1)^{r-1}\binom{r}{r} \\
&= 1 - \left(\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \cdots + (-1)^r\binom{r}{r}\right) \\
&= 1 - \left(\binom{r}{0}1^r(-1)^0 + \binom{r}{1}1^{r-1}(-1)^1 + \binom{r}{2}1^{r-2}(-1)^2 + \cdots + \binom{r}{r}1^0(-1)^r\right) \\
&= 1 - (1-1)^r = 1
\end{aligned}$$

by the Binomial Theorem. Since this holds true for any r, every element of $|A_1 \cap A_2 \cap \cdots \cap A_n|$ is counted exactly once by inclusion-exclusion.

3. A *derangement* is a permutation of a set which leaves no element in its original position. Using the inclusion-exclusion principle, prove that the number of derangements of a set with n elements is

$$D_n = n!\,(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})$$

Solution: Let A_i be the set of permutations of $[n]$ which leave i fixed. Derangements are permutations which leave no element fixed; thus, we're looking for the cardinality of the set $\overline{A_1 \cup \cdots \cup A_n}$, which is $n! - |A_1 \cup \cdots \cup A_n|$. We have $|A_i| = (n-1)!$, $|A_i \cap A_j| = (n-2)!$, $|A_i \cap A_j \cap A_k| = (n-3)!$, and so forth, because if we fix the mappings of r elements then there are $(n-r)!$ ways to arrange the other elements in the permutation. Therefore, the Inclusion-Exclusion Principle gives us

$$\begin{aligned}
&|A_1 \cup \cdots \cup A_n| = \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n-1}(0)! \\
&= \frac{n!}{1!\,(n-1)!}(n-1)! - \frac{n!}{2!\,(n-2)!}(n-2)! + \frac{n!}{3!\,(n-3)!}(n-3)! - \cdots + (-1)^{n-1}\frac{n!}{n!\,0!}0! \\
&= n!\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}\right)
\end{aligned}$$

The formula for D_n follows.

4. *Challenge:* Prove that if $m \leq n$, then the number of onto functions from $[m]$ to $[n]$ is

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m.$$

(*Hint:* Let A_k be the set of functions from $[m]$ to $[n]$ which do not map any element of $[m]$ to k.)
Solution: Define A_k as the hint suggests. Onto functions are those which map to every element of $[n]$; in other words, those which do not belong to any A_k. We're thus looking for the cardinality of $\overline{A_1 \cup \cdots \cup A_n}$, which is $n^m - |A_1 \cup \cdots \cup A_n|$. There are $\binom{n}{r}$ terms in the r'th summation of the inclusion-exclusion formula, and each of those terms is $(n-r)^m$. The formula follows after some algebra.

5. Prove that the Fibonacci sequence $\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ has a closed form defined by $F_n = \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{\sqrt{5}}\beta^n$.
Solution: The Fibonacci sequence obeys the formula $F_{n+2} = F_{n+1} + F_n$, which means it has the closed form $F_n = c\alpha^n + d\beta^n$ for some $c, d \in \mathbb{R}$, as proved in class. We can find c and d by plugging in 0 and 1: $F_0 = 1 = c\alpha^0 + d\beta_0 = c + d$, and $F_1 = 1 = c\alpha + d\beta$. From there, verifying that $c = \frac{1}{\sqrt{5}}$ and $d = \frac{-}{1}\sqrt{5}$ is a matter of algebraic manipulation.

6. The *Lucas numbers* are defined by $\{L_n\} = \{1, 3, 4, 7, 11, 18, \ldots\}$. What is their closed form?
Solution: Again, we have a sequence with the form $L_{n+2} = L_{n+1} + L_n$, so $L_n = c\alpha^n + d\beta^n$ for some c, d. Taking the same approach as in problem 5, we have $c + d = 1$ and $c\alpha + d\beta = 3$. If you substitute and solve, you'll find that $c = \alpha$ and $d = \beta$, so $L_n = \alpha^{n+1} + \beta^{n+1}$.