Worksheet 20: April 10

Principles to Remember

- **Inclusion-Exclusion Principle:** When given a finite union of finite sets, this is how we find its size.
  - $|A \cup B| = |A| + |B| - |A \cap B|$
  - $|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$
  - $|A \cup B \cup C \cup D| = (|A| + |B| + |C| + |D|) - (|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|) + (|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|) - |A \cap B \cap C \cap D|$
  - $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|$
- **Simple recurrence relations:** Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and let $\beta = \dfrac{1-\sqrt{5}}{2} = \dfrac{1}{\alpha}$. If $\{a_n\}$ is a sequence, then $a_{n+2} = a_{n+1} + a_n$ if and only if there exist some $c, d \in \mathbb{R}$ such that $a_n = c\alpha^n + d\beta^n$.

Exercises

1. Draw Venn diagrams to illustrate the Inclusion-Exclusion Principle for unions of two and three sets.

2. Prove the Inclusion-Exclusion Principle. (*Hint:* Consider how many times an element belonging to exactly r of the A_i's is counted in each sum.)

3. A *derangement* is a permutation of a set which leaves no element in its original position. Using the inclusion-exclusion principle, prove that the number of derangements of a set with n elements is

$$D_n = n!\,(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!})$$

4. *Challenge:* Prove that if $m \leq n$, then the number of onto functions from $[m]$ to $[n]$ is

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m.$$

(*Hint:* Let A_k be the set of functions from $[m]$ to $[n]$ which do not map any element of $[m]$ to k.)

5. Prove that the Fibonacci sequence $\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ has a closed form defined by $F_n = \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{\sqrt{5}}\beta^n$.

6. The *Lucas numbers* are defined by $\{L_n\} = \{1, 3, 4, 7, 11, 18, \ldots\}$. What is their closed form?