Worksheet 1: January 22 (Solutions)

1 Propositional Logic

1. Determine whether each of the following sentences is a proposition or not. If it is a proposition, determine its truth value.

   (a) Take BART from Downtown Berkeley to Embarcadero.

   (b) $1 + 1 = 2$.

   (c) San Francisco is the largest city in California.

   (d) $x^2 + y^2 = z^2$.

   (e) This sentence is false.

   Solution: Propositions are *statements* that are either *true or false*. The only propositions among these five are (b), which is true, and (c), which is false. (a) is not a statement, it's an instruction. (d) is not true or false for unspecified x, y, and z, and only becomes true or false when we actually plug in numbers. (e) is neither true or false: if it were true, it would be false, and if it were false, it would be true.

2. Describe the difference between $p \vee q$ and $p \oplus q$. How do these relate to the English word "or"?

   Solution: $p \vee q$ is true if *at least* one of p or q is true. $p \oplus q$ is true if *exactly* one of p or q is true. Though natural language is more ambiguous, we usually use "p or q" to mean $p \vee q$ and "either p or q" to mean $p \oplus q$.

3. Draw a truth table for the following:

   (a) $p \wedge \neg p$

   (b) $p \rightarrow (p \vee q)$

   (c) $(p \rightarrow q) \rightarrow (p \rightarrow r)$

   Solution:

p	$\neg p$
T	F
F	T

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

p	q	r	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

4. Consider the following three propositions:

   (a) $p =$ I go shopping for food
   (b) $q =$ I order at Cheeseboard
   (c) $r =$ I make dinner

   Rephrase the following English sentences using logical operators:

   (a) If I go shopping for food, then I make dinner.
   (b) If I don't go shopping for food, then I order at Cheeseboard.
   (c) I either make dinner or order at Cheeseboard.

   Write each English sentence above using different wording, so that each retains the same meaning when expressed using logical operators.

   Solution:

   (a) $p \to r$. I do not go shopping or I make dinner.
   (b) $\neg p \to q$. I go shopping or I order at cheeseboard.
   (c) $r \oplus q$. I make dinner or order at Cheeseboard, but not both.

5. Show that the converse and the inverse of a conditional statement are equivalent.

   Solution: Given a conditional statement $p \to q$, its converse is $q \to p$, and its inverse is $\neg p \to \neg q$.

p	q	$q \to p$	$\neg p \to \neg q$	$(q \to p) \equiv (\neg p \to \neg q)$
T	T	T	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

2 Propositional Equivalences

6. Use De Morgan's laws to find the negation of each of the following statements:

   (a) Kwame will take a job in industry or go to graduate school.

   (b) Yoshiko knows Java and calculus.

   (c) Celia has been to a Giants game and an A's game.

   Solution:

   - Kwame will not take a job in industry, and he will not go to graduate school.
   - Yoshiko doesn't know Java or doesn't know calculus.
   - Celia hasn't been to a Giants game or hasn't been to an A's game.

7. Use a truth table to verify the distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

   Solution:

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

8. Determine whether each of these compound propositions is satisfiable.

   (a) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

   (b) $(p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg s) \wedge (p \vee \neg s \vee \neg r) \wedge (\neg p \vee \neg q \vee \neg s) \wedge (p \vee q \vee \neg s)$

   Solution: (a) is not satisfiable: you can verify this with a truth table. (b) is satisfiable: set p true and q false.

9. How many different truth tables of compound propositions are there that involve the propositional variables p and q?

   Solution: $2^4 = 16$

10. The n-Queens problem (see section 1.3.6 in the textbook). The n-queens problem asks for an arrangement of n queens on an $n \times n$ chessboard so that no queen can attack another queen (no two queens can be placed in the same row, column, or diagonal). Introduce variables $p(i, j)$ for each location (i, j) in the chessboard, such that $p(i, j)$ is true is there is a queen at (i, j) and false otherwise.

    In row i, there exists a queen can be expressed as $\bigvee_{j=1}^{n} p(i, j)$. Then as there is a queen in each row, that can be expressed as

    $$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} p(i, j).$$

    That row $i = 1$ has no more than one queen can be expressed as $\bigwedge_{j=1}^{n-1} \bigwedge_{k=j+1}^{n} \neg p(i, j) \vee \neg p(i, k)$. Then as there is no more than one queen in each row, that can be expressed as

    $$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n-1} \bigwedge_{k=j+1}^{n} \neg p(i, j) \vee \neg p(i, k).$$

    Write similar assertions that each column and diagonal must contain at least and no more than one queen. Compute the number of possible solutions to the n-queens problem when $n = 2, 3, 4$.

    Solution: The assertion that eacn column contains at least one queen can be written as

    $$\bigwedge_{j=1}^{n} \bigvee_{i=1}^{n} p(i, j).$$

    The assertion that each column must contain no more than one queen can be written as

    $$\bigwedge_{j=1}^{n} \bigwedge_{i=1}^{n-1} \bigwedge_{k=i+1}^{n} \neg p(i, j) \vee \neg p(k, j).$$

    The assertions for the diagonals are significantly more annoying for little added conceptual benefit, so I'm not going to do them.