Worksheet 19: April 8 (Solutions)

Principles to Remember

- Remember the following definitions: **experiment**, **sample space**, **event**, **probability distribution**, **random variable**, **expected value**, **independence**.
- **Variance:** The variance of a random variable X defined on a sample space S is defined as $V(X) = \sum_{s\in S} p(s)(X(s)-E(X))^2$, which is equivalent to $E((X-E(X))^2)$ *(why?)*.
- Properties of variance:
  1. $V(aX) = a^2V(X)$ for any $a \in \mathbb{R}$ and any random variable X
  2. $V(X + a) = V(X)$ for any $a \in \mathbb{R}$ and any random variable X
  3. $V(X + Y) = V(X) + V(Y)$ for ***independent*** random variables X and Y
- **Chebyshev's inequality:** If X is a random variable on a sample space S, and r is a positive real number, then $p(|X(s) - E(X)| \geq r) \leq V(X)/r^2$.

Exercises

1. Let X be the value that comes up when a fair die is rolled. Find $V(X)$.
   Solution: $E(X) = \frac{7}{2}$, so $V(X) = \frac{1}{6}((1-\frac{7}{2})^2+(2-\frac{7}{2})^2+(3-\frac{7}{2})^2+(4-\frac{7}{2})^2+(5-\frac{7}{2})^2+(6-\frac{7}{2})^2) = \frac{1}{6}((\frac{5}{2})^2+(\frac{3}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2+(\frac{3}{2})^2+(\frac{5}{2})^2) = \frac{1}{6}(\frac{25+9+1+1+9+25}{4}) = \frac{70}{24} = \frac{35}{12}$
2. What is the variance of the number of heads that come up when a fair coin is flipped ten times?
   Solution: Let X_i be 1 if the i-th coin is heads and 0 if it's tails; then we're looking for $V(X_1 + \cdots + X_{10})$. Because the coin flips are independent, this is equal to $V(X_1) + \cdots + V(X_{10})$, and because the X_i's are identically distributed, this is equal to $10V(X_1)$. We have $E(X_1) = \frac{1}{2}$, so $V(X_1) = \frac{1}{2}(0-\frac{1}{2})^2 + \frac{1}{2}(1-\frac{1}{2})^2 = \frac{1}{4}$. The answer is therefore $\frac{10}{4} = \frac{5}{2}$.
3. Prove $V(X) = E(X^2) - E(X)^2$. (This formula is often a more convenient way of finding the variance; know it!)
   Proof: $V(X) = E((X-E(X))^2) = E(X^2 - 2XE(X) + E(X)^2)$. By the linearity of expectation and the fact that $E(X)$ is a real number, this is equal to $E(X^2) - 2E(X)E(X) + E(X)^2$, which is equal to $E(X^2) - E(X)^2$.

4. Prove the three properties of variance listed above in "Principles to Remember."
   Property 1: $V(aX) = E((aX)^2) - E(aX)^2 = E(a^2X^2) - (aE(X))^2 = a^2E(X^2 - a^2E(X) = a^2V(X)$
   Property 2: $V(X+a) = E((X+a-E(X+a))^2 = E((X+a-E(X)-a)^2) = E((X-E(X))^2) = V(X)$
   Property 3: $V(X+Y) = E((X+Y)^2) - E(X+Y)^2 = E(X^2+2XY+Y^2) - (E(X)+E(Y))^2 = E(X^2)+2E(X)E(Y)+E(Y^2)-(E(X)^2+2E(X)E(Y)+E(Y)^2) = (E(X^2)-E(X)^2)+(E(Y^2)-E(Y)^2) = V(X)+V(Y)$

5. Prove that independence is required for the third property: Describe two non-independent random variables X and Y such that $V(X+Y) \neq V(X)+V(Y)$.
   Solution: Let X be 1 if a coin comes up heads and 0 if it comes up tails, and let $Y = X$; that is, X and Y will either both be 1 or both be 0. Then $V(X+Y) = V(2X) = 4V(X) = 1$, but $V(X)+V(Y) = 2V(X) = \frac{1}{2}$.

6. Let F be the event that when flipping n coins, the number of tails that comes up deviates from the mean by more than $5\sqrt{n}$. Use Chebyshev's Inequality to find an upper bound on $p(F)$.
   Solution: Let X be the number of tails; then $V(X) = \frac{n}{4}$, from our solution to problem 2. From Chebyshev, $p(F) = p(|X(s)-E(X)| \geq 5\sqrt{n}) \leq V(X)/(5\sqrt{n})^2 = \frac{n}{4} \cdot \frac{1}{25n} = \frac{4}{25}$.

7. Suppose that a recycling center recycles an average of 50,000 aluminum cans a day, with a variance of 10,000 cans. Use Chebyshev's Inequality to provide a lower bound on the probability that the center will recycle between 40,000 and 60,000 cans on a certain day.
   Solution: Let X be the number of cans recycled; then $p(40000 \leq X \leq 60000) = p(|X-E(X)| \leq 10000) = 1 - p(|X-E(X)| > 10000) \geq 1 - V(X)/(10000)^2 = 1 - 10000/(10000)^2 = 0.9999$