Worksheet 18: April 3 (Solutions)

Principles to Remember

- Remember what an **experiment**, **sample space**, **event**, and **probability distribution** are.
- Know the following terms:
  - **random variable:** A *function* from a sample space to $\mathbb{R}$.
  - **expected value:** The EV of a random variable X on a sample space S is defined as $E(X) = \sum_{s \in S} p(s)X(s)$.
- **Independence:** Two RV's X and Y on S are independent if

$$p(X = s \wedge Y = s) = p(X = s)p(Y = s) \quad \forall s \in S.$$

- Linearity of expected value:
  - $E(X + Y) = E(X) + E(Y)$ for **any** random variables X and Y
  - $E(XY) = E(X)E(Y)$ for **independent** random variables X and Y
  - $E(aX + b) = aE(X) + b$ for $a, b \in \mathbb{R}$ and any random variable X

Exercises

1. Consider rolling a fair die 4 times.
   (a) What is the expected number of 6's that appear?
   Solution: $4 \times \frac{1}{6} = \frac{4}{6}$
   (b) What is the expected sum of the numbers that appear?
   Solution: $4 \times \frac{1+2+3+4+5+6}{6} = 4 \times \frac{21}{6} = 14$
2. Repeat the previous problem, but now consider a biased die, where 6 comes up twice as often as any other number.
   Solution: The expected number of 6's that appear is $4 \times \frac{2}{7} = \frac{8}{7}$. The expected sum of the numbers that appear is $4 \times \frac{1+2+3+4+5+2(6)}{7} = 4 \times \frac{27}{7} = \frac{108}{7}$.

3. Suppose that we flip a fair coin until a tails comes up or we flip it 5 times. What is the expected number of times that we flip the coin?
Solution: Let X be the number of times we flip the coin. Then $p(X=1)=\frac{1}{2}$, $p(X=2)=\frac{1}{4}$, $p(X=3)=\frac{1}{8}$, $p(X=4)=\frac{1}{16}$, and $p(X=5)=\frac{1}{16}$. The expected value of X is therefore $E(X)=\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 3+\frac{1}{16}\cdot 4+\frac{1}{16}\cdot 5=\frac{31}{16}$.

4. Suppose that we flip a fair coin until a tails comes up. What is the expected number of times that we flip the coin?
Solution: Let X be the number of times we flip the coin. This time, $p(X=n)=\frac{1}{2^n}$ for all $n \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned} E(X) &= \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \\ &\quad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \\ &\quad\quad + \frac{1}{8} + \frac{1}{16} + \cdots \\ &\quad\quad\quad + \frac{1}{16} + \cdots \\ &\quad\quad\quad\quad + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 2 \end{aligned}$$

5. The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple choice questions, each worth 4 points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple choice question correctly is 0.8. What is her expected score on the final?
Solution: $50\cdot 0.9+25\cdot 0.8=65$

6. Suppose X and Y are random variables in a sample space S, and $X(s)$ and $Y(s)$ are nonnegative for all $s \in S$. Let Z be the random variable defined by $Z(s)=\max\{X(s),Y(s)\}$. Show that $E(Z)\le E(X)+E(Y)$.
Solution: $E(Z)=\sum_{s\in S}p(s)Z(s)\le\sum_{s\in S}p(s)(X(s)+Y(s))=\left(\sum_{s\in S}p(s)X(s)\right)+\left(\sum_{s\in S}p(s)Y(s)\right)=E(X)+E(Y)$