Worksheet 17: April 1

Principles to Remember

- Remember the following terms relating to discrete probability spaces:
  - **experiment:** A procedure that yields one of a given set of possible outcomes
  - **sample space:** The set S of possible outcomes, which (in discrete probability) is finite or countably infinite
  - **event:** A subset of S
  - **probability distribution:** A function $p : S \to [0,1]$ such that $\sum_{s\in S} p(s) = 1$. We say that $p(s)$ is the **probability** of an outcome $s \in S$. The probability of an event $E \subseteq S$ is defined as $p(E) = \sum_{s\in E} p(s)$.
- **Conditional probability:** If E and F are two events, the *probability of E given F* is defined as $p(E \mid F) = \dfrac{p(E\cap F)}{p(F)}$.
- **Independence:** Two events E and F are *independent* if $p(E \cap F) = p(E)p(F)$.
- **Bayes' Theorem:** If $p(E) \neq 0$ and $p(F) \neq 0$, then

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}.$$

Exercises

1. Let E and F be events such that $p(E) = 0.2$ and $p(F) = 0.7$. Find the largest and smallest possible values of...
   (a) $p(E \cup F)$
   (b) $p(E \cap F)$
   (c) $p(E \mid F)$
   (d) $p(F \mid E)$
2. If E and F are disjoint and independent, what can you say about their probabilities?
3. Show that if E and F are events, then $p(E \cap F) \geq p(E) + p(F) - 1$. This is known as Bonferroni's inequality.

4. Prove Bayes' Theorem.

5. Suppose $p(E) = 1/3$, $p(F) = 1/2$, and $p(E \mid F) = 2/5$. Find $p(F \mid E)$.

6. I have two boxes. The first box contains three red balls and two blue balls. The second box contains one red ball and five blue balls. If I reach into a box at random and pull out a red ball, what is the probability that I reached into the first box?

7. Suppose that a test for opium use has a 2% false positive rate and a 5% false negative rate. That is, 2% of people who do not use opium test positive for opium, and 5% of opium users test negative for opium. Furthermore, suppose that 1% of people actually use opium.

   (a) Find the probability that someone who tests negative for opium use does not use opium.

   (b) Find the probability that someone who tests positive for opium use actually uses opium.