Worksheet 16: March 18 (Solutions)

Principles to Remember

- Know the following terms relating to discrete probability spaces:
  - **experiment:** A procedure that yields one of a given set of possible outcomes
  - **sample space:** The set S of possible outcomes, which (in discrete probability) is finite or countably infinite
  - **event:** A subset of S
  - **probability distribution:** A function $p : S \to [0, 1]$ such that $\sum_{s \in S} p(s) = 1$. We say that $p(s)$ is the **probability** of an outcome $s \in S$.
- **Uniform probability:** If S is finite and all outcomes are equally likely, then p is the *uniform probability distribution*

$$p(s) = \frac{1}{|S|} \quad \text{for all } s \in S$$

- **Probability of events:** If $E \subseteq S$, then we define $p(E) = \sum_{s \in E} p(s)$. In particular, if p is the uniform probability distribution, then

$$p(E) = \frac{|E|}{|S|}.$$

Exercises

1. What is the probability that...
   (a) a fair die will come up six when it is rolled?
   Solution: $\frac{1}{6}$
   (b) a card selected at random from a standard 52-card deck will be an ace?
   Solution: $\frac{4}{52} = \frac{1}{13}$
   (c) a randomly selected day in 2024 will be in February?
   Solution: $\frac{29}{366}$
   (d) a card selected at random from a standard 52-card deck will be a king or a heart?
   Solution: $\frac{4 + 13 - 1}{52} = \frac{16}{52} = \frac{4}{13}$

2. What is the probability that a random five-card poker hand...

   (a) is a royal flush (a ten, jack, queen, king, and ace, all of the same suit)?
   Solution: $\frac{4}{\binom{52}{5}} = \frac{4}{2598960}$

   (b) contains at least one ace?
   Solution: $\frac{\binom{52}{5} - \binom{48}{5}}{\binom{52}{5}}$

   (c) contains exactly one ace?
   Solution: $\frac{4 \cdot \binom{48}{5}}{\binom{52}{5}}$

   (d) contains cards of five different ranks?
   Solution: There are $\binom{13}{5}$ ways to choose the ranks of the cards, and 4 ways to choose the suit of each card. The answer is $\frac{\binom{13}{5} \cdot 4^5}{\binom{52}{5}}$.

   (e) contains two pairs (two each of two different ranks, and one more of a third rank)?
   Solution: There are $\binom{13}{2}$ ways to choose the ranks of the pairs. Once we've done so, there are $\binom{4}{2}$ ways to choose the suits in the first pair, $\binom{4}{2}$ ways to choose the suits in the second pair, 11 ways to choose the rank of the lone card, and 4 ways to choose the suit of the lone card. The answer is $\frac{\binom{13}{2} \cdot \binom{4}{2}^2 \cdot 11 \cdot 4}{\binom{52}{4}}$.

3. What is the probability that a random positive integer not exceeding 100 is...

   (a) divisible by 3?
   Solution: $\frac{33}{100}$

   (b) divisible by 5 or 7?
   Solution: $\frac{20 + 14 - 2}{100} = \frac{32}{100} = \frac{8}{25}$

4. Which is more likely: rolling a total of 5 when two dice are rolled, or rolling a total of 5 when three dice are rolled?
   Solution: When two dice are rolled, the combinations that sum to 5 are $(1, 4)$, $(2, 3)$, $(3, 2)$, and $(4, 1)$, so the probability of rolling a total of 5 is $\frac{4}{36}$. When three dice are rolled, the combinations that sum to 5 are $(1, 1, 3)$, $(1, 3, 1)$, $(3, 1, 1)$, $(1, 2, 2)$, $(2, 1, 2)$, and $(2, 2, 1)$, so the probability of rolling a total of 5 is $\frac{6}{216}$. The probability is greater for two dice.

5. Suppose that a contest is held where winners are selected at random for first, second, and third prize. If Alice, Bill, and Charlie enter the contest, and there are 200 participants total, what is the probability that Alice wins first prize, Bill wins second prize, and Charlie wins third prize, if

   (a) no one can win more than one prize?

   Solution: $\dfrac{1}{200 \cdot 199 \cdot 198}$

   (b) winning more than one prize is allowed?

   Solution: $\dfrac{1}{200^3}$

6. What is the probability of rolling at least one six when a fair die is rolled four times?

   Solution: $\dfrac{6^3 - 5^3}{6^3}$