Worksheet 15: March 13 (Solutions)

Principles to Remember

- **Permutations with Repetitions:** The number of r-permutations from a set with n elements when repetition is allowed is n^r.
- **Generalized Permutations:** If we have n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, and $n = n_1 + n_2 + \cdots + n_k$, then the number of permutations of all these objects is $\frac{n!}{n_1!\,n_2!\cdots n_k!}$. This is known as the **multinomial coefficient** $\binom{n}{n_1, n_2, \ldots, n_k}$.
- **Combinations with Repetitions:** The number of r-combinations from a set with n elements when repetition is allowed is $\binom{n+r-1}{r}$.

Exercises

1. How many permutations are there of the word 'COMBINATORICS'?
   Solution: The letters 'C', 'O', and 'I' each appear twice, and all other letters appear once, so the solution is $\frac{13!}{2!\,2!\,2!}$
2. How many ways are there to line up three apples, four bananas, five oranges, and six kiwis?
   Solution: $\binom{18}{3,4,5,6} = \frac{18!}{3!\,4!\,5!\,6!}$
3. How many strings of 10 ternary digits (0, 1, or 2) contain exactly three 0's, five 1's, and two 2's?
   Solution: $\binom{10}{3,5,2} = \frac{10!}{3!\,5!\,2!}$
4. How many solutions are there to the equation $x_1+x_2+x_3+x_4 = 11$, where x_1, x_2, x_3, x_4 are...

   (a) nonnegative integers?
   Solution: This is a bars-and-stars type of problem; see section 6.5.3 in your textbook. The answer is $\binom{11+4-1}{4-1} = \binom{14}{3}$.

   (b) positive integers?
   Solution: Let $y_1 = x_1 - 1$, $y_2 = x_2 - 1$, $y_3 = x_3 - 1$, and $y_4 = x_4 - 1$. This

problem is equivalently asking for the number of solutions to $y_1+y_2+y_3+y_4=7$ in nonnegative integers, which is $\binom{7+4-1}{4-1}=\binom{10}{3}$.

(c) integers?
Solution: Infinite.

5. Given a standard 52-card deck and four players, how many ways are there to deal a hand of 13 cards to each player?
Solution: $\binom{52}{13,13,13,13}=\dfrac{52!}{13!\,13!\,13!\,13!}$

6. How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?
Solution: The fact that all coins of the same denomination are identical is relevant here. Our chosen coins can consist of 8 pennies and 0 nickels, or of 7 pennies and 1 nickel, or of 6 pennies and 2 nickels, etc., down to 0 pennies and 8 nickels. There are 9 total possibilities.

7. How many ways are there to distribute...

(a) n distinguishable balls into k distinguishable boxes?
Solution: n^k

(b) n indistinguishable balls into k distinguishable boxes?
Solution: $\binom{n+k-1}{k-1}$

8. How many ways can n books be placed on k distinguishable shelves...

(a) if the books are n identical copies of Kenneth H. Rosen's *Discrete Mathematics and Its Applications*?
Solution: $\binom{n+k-1}{k-1}$

(b) if no two books are the same, but the positions of the books on the shelves don't matter (only which shelf they're placed on)?
Solution: k^n

(c) if no two books are the same, and the positions of the books on the shelves matter?
Solution: $n!\cdot\binom{n+k-1}{k-1}$