Worksheet 14: March 11 (Solutions)

Principles to Remember

- **Binomial Theorem:** For any integer $n \geq 0$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

- **Pascal's Identity:** For any integers n and k such that $0 \leq k \leq n$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

1 Binomial Coefficients and Identities

1. Expand the following expressions:
   (a) $(x+y)^5$
   Solution: $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
   (b) $(x^3+y^2)^4$
   Solution: $x^{12} + 4x^9y^2 + 6x^6y^4 + 4x^3y^6 + y^8$

2. Find the coefficient of $x^a y^b$ in the expansion of $(5x^2 - 2y^3)^6$, where...
   (a) $a = 6$, $b = 9$
   Solution: $\binom{6}{3} 5^3 (-2)^3 = -20000$
   (b) $a = 2$, $b = 15$
   Solution: $\binom{6}{5} 5^1 (-2)^5 = -960$
   (c) $a = 10$, $b = 6$
   Solution: 0

3. Prove that $\sum_{j=0}^{n} 3^j \binom{n}{j} = 4^n$.

   Solution: Note that the sum is equal to $\sum_{j=0}^{n} 1^{n-j} 3^j \binom{n}{j}$, which, by the Binomal Theorem, is equal to $(1+3)^n$.

4. Prove Pascal's Identity...

   (a) using a combinatorial proof.
   Solution: Combinatorial proofs require us to count the same thing in two different ways. In this case, consider the number of subsets of size k of a set S, where $|S| = n$. Choose some element $c \in S$; each subset of S either contains c or does not contain c. If a subset of size k contains c, then it has $k-1$ elements which are not c, drawn from the $n-1$ elements in $S \setminus \{c\}$; there are $\binom{n-1}{k-1}$ such subsets. If a subset of size k does not contain c, then all of its k elements are drawn from the $n-1$ elements in $S \setminus \{c\}$; there are $\binom{n-1}{k}$ such subsets. Thus there are a total of $\binom{n-1}{k-1} + \binom{n-1}{k}$ subsets of S that have size k. We also know that there are $\binom{n}{k}$ subsets of S that have size k, so $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

   (b) using an algebraic proof.
   Solution:

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= k \cdot \frac{(n-1)!}{k!(n-k)!} + (n-k) \cdot \frac{(n-1)!}{k!(n-k)!} \\
&= n \cdot \frac{(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} = \binom{n}{k}
\end{aligned}
$$

5. Prove that for any integers n and k such that $1 \leq k \leq n$, it holds that $k\binom{n}{k} = n\binom{n-1}{k-1}$...

   (a) using a combinatorial proof. (*Hint:* Given a set of n elements, show that both sides count the number of ways to select a subset of k elements and then choose one element from among that subset.)
   Solution: As the hint suggests. If we pick a subset of k elements from a set of n elements, there are $\binom{n}{k}$ ways to do so; and if we subsequently mark one element from among that subset, there are k ways to do so, for a total of $k\binom{n}{k}$ ways. If we mark an element first, there are n ways to do so; and if we pick a subset of k elements which contains the marked element, its remaining $k-1$ elements will come from the $n-1$ unmarked elements, so there are $\binom{n-1}{k-1}$ ways to do so, for a total of $n\binom{n-1}{k-1}$ ways.

(b) using an algebraic proof.
Solution:

$$\begin{aligned} k\binom{n}{k} &= k \cdot \frac{n!}{k!(n-k)!} \\ &= nk \cdot \frac{(n-1)!}{k!(n-k)!} \\ &= n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1} \end{aligned}$$

6. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$...

(a) using a combinatorial proof.
Solution: Consider the number of ways to pick a subset of size 2 from an ordered set S of size $2n$; there are $\binom{2n}{2}$ ways to do this. If both elements in the subset are in the first n elements of S, there are $\binom{n}{2}$ ways to choose them; if both elements in the subset are in the last n elements of S, there are $\binom{n}{2}$ ways to choose them; and if one element is in the first n and the other is in the last n, there are n ways to choose each, for a total of n^2 ways. These cases are exhaustive, so $\binom{2n}{2} = \binom{n}{2} + \binom{n}{2} + n^2 = 2\binom{n}{2} + n^2$.

(b) using an algebraic proof.
Solution:

$$\binom{2n}{2} = \frac{(2n!)}{2!(2n-2)!} = \frac{2n(2n-1)}{2} = n(2n-1) = 2n^2 - n$$

$$2\binom{n}{2} + n^2 = 2 \cdot \frac{n!}{2!(n-2)!} + n^2 = 2 \cdot \frac{n(n-1)}{2} + n^2 = 2n^2 - n$$