Worksheet 14: March 11

Principles to Remember

- **Binomial Theorem:** For any integer $n \geq 0$

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n\end{aligned}$$

- **Pascal's Identity:** For any integers n and k such that $0 \leq k \leq n$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

1 Binomial Coefficients and Identities

1. Expand the following expressions:
   (a) $(x+y)^5$
   (b) $(x^3+y^2)^4$

2. Find the coefficient of $x^a y^b$ in the expansion of $(5x^2 - 2y^3)^6$, where...
   (a) $a = 6$, $b = 9$
   (b) $a = 2$, $b = 15$
   (c) $a = 10$, $b = 6$

3. Prove that $\displaystyle\sum_{j=0}^{n} 3^j \binom{n}{j} = 4^n$.

4. Prove Pascal's Identity...

   (a) using a combinatorial proof.

   (b) using an algebraic proof.

5. Prove that for any integers n and k such that $1 \leq k \leq n$, it holds that $k\binom{n}{k} = n\binom{n-1}{k-1}$...

   (a) using a combinatorial proof. (*Hint:* Given a set of n elements, show that both sides count the number of ways to select a subset of k elements and then choose one element from among that subset.)

   (b) using an algebraic proof.

6. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$...

   (a) using a combinatorial proof.

   (b) using an algebraic proof.