Worksheet 13: March 6 (Solutions)

Principles to Remember

- **Pigeonhole Principle:** Putting $n+1$ objects into n boxes always results in one box with at least two objects in it. More generally, placing k objects into n boxes always results in one box with at least $\lceil k/n \rceil$ objects in it.
- An ***r*-permutation** of a set S is an *ordered* arrangement of r elements of S. If S has n elements, its number of r-permutations is $\frac{n!}{(n-r)!}$.
- An ***r*-combination** of a set S is an *unordered* arrangement of r elements of S. If S has n elements, its number of r-combinations is $\frac{n!}{(n-r)!r!}$, which we denote as $\binom{n}{r}$ and pronounce "n choose r."

1 Pigeonhole Principle

1. Suppose you have 3 spheres and 7 cubes, each labelled with a number between 0 and 9. Show that there are at least two different sphere-cube pairs whose sums are equal. Is this still true with 6 cubes instead of 7?
   Solution: The minimum sum of a sphere-cube pair is $0+0=0$, and the maximum sum is $9+9=18$, so there are 19 possible sums. Since we have $3\times 7=21$ pairs, at least two pairs must have the same sum by the pigeonhole principle. The pigeonhole principle doesn't tell us anything for the case of 6 cubes, since there are only 18 pairs in that case. (It's still true, though, but much harder to prove.)

2. What is the smallest integer n such that any subset of $\{1,2,\ldots,9\}$ with n elements is guaranteed to contain two numbers adding to 10?
   Solution: Consider the five "boxes" $\{1,9\}$, $\{2,8\}$, $\{3,7\}$, $\{4,6\}$, and $\{5\}$. If we pick any six numbers from [9], then two of them must fall in the same box, and thus some to 10. This is a strict lower bound on n, since $\{1,2,3,4,5\}$ contains no two elements that sum to 10. Therefore $n=6$.

3. Show that in a group of 6 people, where any two people are either enemies or friends, there are either three mutual friends or three mutual enemies.
   Solution: Select any one person, whom we'll call A. There are five other people, so by the pigeonhole principle, either three of them are friends with A, or three of them are enemies with A. Without loss of generality, assume that A has three friends, whom we'll call B, C, and D. If B and C are friends, then A,B,C form a mutual friends triple; if B and D are friends, then A,B,D form a mutual friends triple; if C and D are friends, then A,C,D form a mutual friends triple; and if none of these cases hold, then B, C, and D are all enemies with each other, so B,C,D

form a mutual enemies triple. Solutions to the generalized "party problem" are called *Ramsey numbers*; 6 is the Ramsey number $R(3,3)$.

4. Suppose there is a hotel with infinitely many rooms, numbered $1, 2, 3, \ldots$, and infinitely many guests, numbered $0, 1, 2, 3, \ldots$. Does the pigeonhole principle imply that some room has at least two guests?
Solution: Nope – the pigeonhole principle only applies to finite sets. Countable sets can be mapped bijectively to each other, as we have seen in previous units.

2 Permutations and Combinations

5. How many ways are there for three puffins and six penguins to stand in a line such that...

   (a) ...all puffins stand together?
   Solution: Treating the puffins as a block and each individual penguin as a block implies there are 7! ways to arrange the blocks, and there are 3! ways to arrange the puffins in their block, so the answer is $7! \cdot 3!$.

   (b) ...all penguins stand together?
   Solution: Now we have 4 blocks, one of which contains 6 elements, so the answer is $4! \cdot 6!$.

6. How many permutations of the string 'ABCDEFG'...

   (a) ...contain both 'ABC' and 'DE' as consecutive substrings?
   Solution: The four blocks are 'ABC', 'DE', 'F', and 'G'. The answer is 4!.

   (b) ...have 'A' anywhere before 'B'?
   Solution: All permutations either have A before B or have B before A, and there are equally many permutations of each type, so the answer is $\frac{7!}{2}$.

7. Find a formula for the number of ways to seat n people around a circular table, where seatings are considered the same if every person has the same two neighbors (without regard to which side those neighbors are sitting on).
Solution: We're counting *equivalency classes* of permutations, where each class has size $2n$ because of rotation and reflection. Since each permutation belongs to such a class, there are $\frac{n!}{2n} = \frac{(n-1)!}{2}$ equivalency classes.

8. How many ways are there for a horse race with four horses to finish if ties are possible? Note that any number of horses may tie in any position.
Solution: We need to consider several cases here.

   - If no horses tie, there are $4! = 24$ ways to arrange the horses.
   - If two horses tie, there are $\binom{4}{2}$ ways to form the pair and 3! ways to arrange the three blocks, for a total of 36 ways.

- If three horses tie, there are $\binom{4}{3}$ ways to form the triple and 2! ways to arrange the two blocks, for a total of 8 ways.
- If two pairs of horses each tie, there are 3 ways to form the pairs and 2 ways to arrange them, for a total of 6 ways.
- Finally, all four horses tying is 1 additional way the race may go.

The total number of possible outcomes is $24 + 36 + 8 + 6 + 1 = 75$.

9. Prove that $\displaystyle\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ for $n, r \in \mathbb{N}$ and $n > r$. (This is known as the *hockey-stick identity.* Why?)
Solution: You can prove this either combinatorially or via induction on n; I'm going to do the latter. In the base case $n = r$, we have $\binom{r}{r} = \binom{r+1}{r+1}$, which is true because they're both equal to 1. For the inductive hypothesis, we assume $\sum_{i=r}^{n-1} \binom{i}{r} = \binom{n}{r+1}$ for some $n > r$. In the inductive step, we see that

$$\sum_{i=r}^{n} \binom{i}{r} = \left(\sum_{i=r}^{n-1} \binom{i}{r}\right) + \binom{n}{r} = \binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1}.$$

The last equality is from Pascal's rule.

This is known as the hockey-stick identity because if you trace out the coefficients it deals with on Pascal's triangle, they form a long line that zags for one step at the end, so it looks a bit like a hockey stick.