Worksheet 13: March 6

Principles to Remember

- **Pigeonhole Principle:** Putting $n+1$ objects into n boxes always results in one box with at least two objects in it. More generally, placing k objects into n boxes always results in one box with at least $\lceil k/n \rceil$ objects in it.
- An ***r*-permutation** of a set S is an *ordered* arrangement of r elements of S. If S has n elements, its number of r-permutations is $\frac{n!}{(n-r)!}$.
- An ***r*-combination** of a set S is an *unordered* arrangement of r elements of S. If S has n elements, its number of r-combinations is $\frac{n!}{(n-r)!r!}$, which we denote as $\binom{n}{r}$ and pronounce "n choose r."

1 Pigeonhole Principle

1. Suppose you have 3 spheres and 7 cubes, each labelled with a number between 0 and 9. Show that there are at least two different sphere-cube pairs whose sums are equal. Is this still true with 6 cubes instead of 7?

2. What is the smallest integer n such that any subset of $\{1, 2, \ldots, 9\}$ with n elements is guaranteed to contain two numbers adding to 10?

3. Show that in a group of 6 people, where any two people are either enemies or friends, there are either three mutual friends or three mutual enemies.

4. Suppose there is a hotel with infinitely many rooms, numbered $1, 2, 3, \ldots$, and infinitely many guests, numbered $0, 1, 2, 3, \ldots$. Does the pigeonhole principle imply that some room has at least two guests?

2 Permutations and Combinations

5. How many ways are there for three puffins and six penguins to stand in a line such that...
   (a) ...all puffins stand together?
   (b) ...all penguins stand together?

6. How many permutations of the string 'ABCDEFG'...
   (a) ...contain both 'ABC' and 'DE' as consecutive substrings?
   (b) ...have 'A' anywhere before 'B'?

7. Find a formula for the number of ways to seat n people around a circular table, where seatings are considered the same if every person has the same two neighbors (without regard to which side those neighbors are sitting on).

8. How many ways are there for a horse race with four horses to finish if ties are possible? Note that any number of horses may tie in any position.

9. Prove that $\displaystyle\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ for $n, r \in \mathbb{N}$ and $n > r$. (This is known as the *hockey-stick identity.* Why?)