Worksheet 12: March 4 (Solutions)

1 Counting

Principles to Remember

- Product Rule: A sequence of two tasks, one of which can be performed in n_1 ways and the other of which can be performed in n_2 ways, can be performed in a total of $n_1 n_2$ ways.
- Sum Rule: A single task, which can be performed in n_1 ways or in n_2 other ways, can be performed in a total of $n_1 + n_2$ ways.
- Inclusion-Exclusion: A single task, which can be performed in n_1 ways or n_2 ways (not necessarily distinct), where v ways are counted twice, can be performed in $n_1 + n_2 - v$ ways.

Exercises

1. How many license plates can be formed from seven characters, where each character is either a letter or a number?
2. Let $S = [10]$ (shorthand for the set $\{1, 2, \ldots, 10\}$).
   (a) How many subsets of S are there?
   Answer: 2^{10}
   (b) How many subsets of S are there containing 1? Containing 10?
   Answer: 2^9 each
   (c) How many subsets of S contain both 1 and 10?
   Answer: 2^8
   (d) How many subsets of S contain at least one of 1 or 10?
   Answer: $2^9 + 2^9 - 2^8 = 2^{10} - 2^8$
   (e) How many subsets of S contain neither 1 nor 10?
   Answer: 2^8
3. How many functions from $[m]$ to $[n]$ are there...
   (a) ...in total?
   Answer: n^m
   (b) ...which are injective (1-1)?
   Answer: $\dfrac{n!}{(n-m)!}$

4. How many three-letter initials can people have if...

   (a) Letters cannot be repeated?
   Answer: $26 \cdot 25 \cdot 24$

   (b) The first initial must be A?
   Answer: 26^2

   (c) Both of the above?
   Answer: $25 \cdot 24$

5. How many positive integers between 500 and 1000...

   (a) ...are divisible by 7?
   Answer: $\lfloor \frac{1000}{7} \rfloor - \lfloor \frac{500}{7} \rfloor = 71$

   (b) ...are divisible by both 7 and 11?
   Answer: $\lfloor \frac{1000}{77} \rfloor - \lfloor \frac{500}{77} \rfloor = 6$

   (c) ...are divisible by 7 but not by 11?
   Answer: $71 - 6 = 65$

   (d) ...have distinct digits?
   Answer: $5 \cdot 9 \cdot 8 = 360$

   (e) ...have distinct digits and are even?
   Answer: Take two separate cases. In the case of numbers with an odd hundreds digit, there are 3 choices for the first digit, then 5 for the last digit, then 8 for the middle digit, for a total of $3 \cdot 5 \cdot 8 = 120$. In the case of numbers with an even hundreds digit, there are 2 choices for the first digit, then 4 for the last digit, then 8 for the middle digit, for a total of $2 \cdot 4 \cdot 8 = 64$. Add the two cases together to make $120 + 64 = 184$.

6. How many injective functions are there from a set with 5 elements to a set with...

   (a) 4 elements?
   Answer: 0

   (b) 5 elements?
   Answer: $5! = 125$

   (c) 6 elements?
   Answer: $\frac{6!}{1!} = 720$

   (d) 7 elements?
   Answer: $\frac{7!}{2!} = 2520$

7. How many subsets of [100] have more than one element?
   Answer: $2^{100} - 100 - 1$ (subtract all the subsets containing one or zero elements)

8. How many palindromes are there among the bit strings of length n?
   Answer: $2^{\lceil n/2 \rceil}$