Worksheet 11: February 28

1 More Induction

1. Prove that for any $n \in \mathbb{Z}^+$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

   Solution: The base case for $n = 1$ is true because $1^3 = \frac{1^2(1+1)^2}{4}$. In the inductive case $n \geq 2$, if we assume that $1^3 + \cdots + (n-1)^3 = \frac{(n-1)^2 n^2}{4}$, we have

$$\begin{aligned} 1^3 + \cdots + n^3 &= (1^3 + \cdots + (n-1)^3) + n^3 \\ &= \frac{(n-1)^2 n^2}{4} + n^3 \\ &= (n^2 - 2n + 1)\frac{n^2}{4} + 4n\frac{n^2}{4} \\ &= (n^2 + 2n + 1)\frac{n^2}{4} \\ &= \frac{n^2(n+1)^2}{4}. \end{aligned}$$

2. Prove inductively that for any $n \in \mathbb{Z}^+$ and $p \in \mathbb{R}\backslash\{1\}$, $1+p+p^2+\cdots+p^n = \frac{p^{n+1}-1}{p-1}$.

   Solution: The base case for $n = 1$ is true because $1 + p = \frac{p^2 - 1}{p - 1}$. In the inductive case $n \geq 2$, if we assume that $1 + p + \cdots + p^{n-1} = \frac{p^n - 1}{p - 1}$, we have

$$\begin{aligned} 1 + p + \cdots + p^n &= (1 + p + \cdots + p^{n-1}) + p^n \\ &= \frac{p^n - 1}{p - 1} + p^n \\ &= \frac{p^n - 1}{p - 1} + \frac{p^{n+1} - p^n}{p - 1} \\ &= \frac{p^{n+1} - 1}{p - 1}. \end{aligned}$$

3. Prove that for any $n \in \mathbb{Z}^+$, $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

   Solution: The base case for $n = 1$ is true because $1 \cdot 1! = (1+1)! - 1$. In the inductive case $n \geq 2$, if we assume that $\sum_{k=1}^{n-1} k \cdot k! = n! - 1$, we have

$$\begin{aligned}
\sum_{k=1}^{n} k \cdot k! &= \left(\sum_{k=1}^{n-1} k \cdot k!\right) + n \cdot n! \\
&= n! - 1 + n \cdot n! \\
&= (n+1) \cdot n! - 1 \\
&= (n+1)! - 1.
\end{aligned}$$

4. Evaluate $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$ for some small values of n. What does it seem like the formula should be? Prove this formula by induction.

   Solution: For $n = 1$, we have $\frac{1}{2}$. For $n = 2$, we have $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$. For $n = 3$, we have $\frac{2}{3} + \frac{1}{12} = \frac{3}{4}$. For $n = 4$, we have $\frac{3}{4} + \frac{1}{20} = \frac{4}{5}$. It seems like the formula should be $\frac{n}{n+1}$. This formula holds for the base case, and if we assume that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$, then

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} &= \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n}\right) + \frac{1}{n(n+1)} \\
&= \frac{n-1}{n} + \frac{1}{n(n+1)} \\
&= \frac{(n-1)(n+1) + 1}{n(n+1)} \\
&= \frac{n^2}{n(n+1)} \\
&= \frac{n}{n+1}.
\end{aligned}$$

5. Prove that 3 divides $n^3 + 2n$ for any positive integer n.

   Solution: The base case for $n = 1$ is true because 3 divides $1^3 + 2$. In the inductive case $n \geq 2$, if we assume that $(n-1)^3 + 2(n-1) = 3k$ for some integer k, we have

$$\begin{aligned}
n^3 + 2n &= n^3 - (n-1)^3 + 2n - 2(n-1) + (n-1)^3 + 2(n-1) \\
&= n^3 - (n^3 - 3n^2 + 3n - 1) + 2n - (2n - 2) + 3k \\
&= 3n^2 - 3n + 1 + 2 + 3k \\
&= 3(n^2 - n + 1 + k), \text{ which is divisible by 3.}
\end{aligned}$$

6. Prove that for $n \geq 30$, a postage of n cents can be made using just 4-cent and 11-cent stamps. (Hint: Use strong induction. What is the base case?)
Solution: The base case is that postages of 30, 31, 32, and 33 cents can be made with just 4-cent and 11-cent stamps: this is true because $30 = 2 \cdot 11 + 2 \cdot 4$, $31 = 1 \cdot 11 + 5 \cdot 4$, $32 = 8 \cdot 4$, and $33 = 3 \cdot 11$. In the inductive case $n \geq 34$, assume that all postages between 30 and $n-1$ cents can be made with just 4-cent and 11-cent stamps. Then $n-4$ cents can be made in this way, so to get n cents, simply add a 4-cent stamp to that combination.

7. What's wrong with the following inductive "proof" that the sum of all positive integers is finite?

   - Let $P(n)$ be the statement "the sum of the first n positive integers is finite."
   - Base case: 1 is finite, so $P(1)$ is true.
   - Inductive hypothesis: $P(n)$ is true.
   - Inductive step: $1 + \cdots + (n+1) = [1 + \cdots + n] + (n+1)$. Using $P(n)$, the first sum is a finite number S. Therefore $S + (n+1)$ is finite, so $P(n+1)$ is true.
   - Therefore the sum of all positive integers is finite.

   Solution: The proof is fine until the very last line. It's true that for any integer n, the first positive n integers are infinite. However, ∞ is not an integer, so we cannot use this proof to conclude that the sum of all positive integers – a.k.a., the sum of the first ∞ integers – is finite.

2 Recursive Definitions

8. Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, \ldots$

   (a) $f(n+1) = f(n) + 2$
   Solution: 3, 5, 7, 9

   (b) $f(n+1) = 3f(n)$
   Solution: 3, 9, 27, 81

   (c) $f(n+1) = 2^{f(n)}$
   Solution: 2, 4, 16, 65536

   (d) $f(n+1) = f(n)^2 + f(n) + 1$
   Solution: 3, 13, 183, 33673

   (e) $f(n+1) = 1 - f(n)$
   Solution: 0, 1, 0, 1

9. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, where f_k is the k-th Fibonacci number.
Solution: The base case for $n = 1$ is true because $f_1^2 = f_1 f_2$. In the inductive case $n \geq 2$, if we assume that $f_1^2 + \cdots + f_{n-1}^2 = f_{n-1} f_n$, then

$$\begin{aligned} f_1^2 + \cdots + f_n^2 &= (f_1^2 + \cdots + f_{n-1}^2) + f_n^2 \\ &= f_{n-1} f_n + f_n^2 \\ &= f_n(f_{n-1} + f_n) \\ &= f_n f_{n+1}. \end{aligned}$$