Worksheet 11: February 28

1 More Induction

1. Prove that for any $n \in \mathbb{Z}^+$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

2. Prove inductively that for any $n \in \mathbb{Z}^+$ and $p \in \mathbb{R}\setminus\{1\}$, $1+p+p^2+\cdots+p^n = \dfrac{p^{n+1}-1}{p-1}$.

3. Prove that for any $n \in \mathbb{Z}^+$, $\displaystyle\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

4. Evaluate $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)}$ for some small values of n. What does it seem like the formula should be? Prove this formula by induction.

5. Prove that 3 divides $n^3 + 2n$ for any positive integer n.

6. Prove that for $n \geq 30$, a postage of n cents can be made using just 4-cent and 11-cent stamps. (Hint: Use strong induction. What is the base case?)

7. What's wrong with the following inductive "proof" that the sum of all positive integers is finite?

   - Let $P(n)$ be the statement "the sum of the first n positive integers is finite."
   - Base case: 1 is finite, so $P(1)$ is true.
   - Inductive hypothesis: $P(n)$ is true.
   - Inductive step: $1 + \cdots + (n+1) = [1 + \cdots + n] + (n+1)$. Using $P(n)$, the first sum is a finite number S. Therefore $S + (n+1)$ is finite, so $P(n+1)$ is true.
   - Therefore the sum of all positive integers is finite.

2 Recursive Definitions

8. Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, \ldots$

   (a) $f(n+1) = f(n) + 2$

   (b) $f(n+1) = 3f(n)$

   (c) $f(n+1) = 2^{f(n)}$

   (d) $f(n+1) = f(n)^2 + f(n) + 1$

   (e) $f(n+1) = 1 - f(n)$

9. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, where f_k is the k-th Fibonacci number.