Chapter 10.?: Random Graphs

Wednesday, August 12

Summary

- *Almost all* graphs have a property Q if the probability that (a random graph on n vertices has property Q) approaches 1 as $n \to \infty$.
- Turan's Theorem: Let G be a graph with n vertices such that G is K_{r+1}-free. Then the number of edges in G is at most $\left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$.
- Also Turan's Theorem: Any graph G = (V, E) contains an independent set of size at least $|V|/(D+1)$, where $D = 2|E|/|V|$ is the average degree of the graph.

Random Graphs

1. (★) Let G be a bipartite graph with $n \geq 3$ vertices, and pick 3 distinct vertices at random. Prove that the probability that all 3 vertices are independent is at least $\frac{1}{4} - \frac{K}{n}$ where K is some constant independent of n.
2. Show that almost all graphs are not trees.
3. (★) Show that almost all graphs have a triangle.
4. Let $T(G)$ be the number of triangles in a graph G. If G has n vertices then what is $E(T(G))$? (Hard) What is $Var(T(G))$?
5. Let $C(G)$ be the number of 4-cycles in a graph G. If G has n vertices then what is $E(C(G))$? (Hard) What is $Var(C(G))$?

Turan's Theorem

1. Show that the two formulations of Turan's theorem are equivalent.
2. (★) Define the Turan graph $T(n, r)$ as follows: partition the vertices into r sets of equal or nearly equal size and connect any pair of vertices that are not in the same set. Prove that $T(n, r)$ does not contain K_{r+1} as a subgraph.
3. Show that $T(n, r)$ has the maximum number of edges of any n-vertex graph not containing K_{r+1}.

Miscellany

1. Define the clique number of a graph, $\omega(G)$, to be the largest m such that K_m is a subgraph of G. Show that $\chi(G) \geq \omega(G)$.
2. Show that $\omega(G) = \alpha(\overline{G})$.