Chapter 10.?: Random Graphs

Wednesday, August 12

Summary

- *Almost all* graphs have a property Q if the probability that (a random graph on n vertices has property Q) approaches 1 as $n \to \infty$.
- Turan's Theorem: Let G be a graph with n vertices such that G is K_{r+1}-free. Then the number of edges in G is at most $\left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$.
- Also Turan's Theorem: Any graph G = (V, E) contains an independent set of size at least $|V|/(D+1)$, where $D = 2|E|/|V|$ is the average degree of the graph.

Random Graphs

1. (★) Let G be a bipartite graph with $n \geq 3$ vertices, and pick 3 distinct vertices at random. Prove that the probability that all 3 vertices are independent is at least $\frac{1}{4} - \frac{K}{n}$ where K is some constant independent of n.

   The probability is minimized if G is complete and has its vertices evenly split between the two sides. (Why?) In this case, the three vertices are independent if and only if they are all on the same side of the bipartite graph. If G has $n = 2m$ vertices, then the chance is $\frac{\binom{m}{3}+\binom{m}{3}}{\binom{2m}{3}}$, which is equal to 1/4 plus some lower order terms that go to zero as $n \to \infty$. If $n = 2m+1$, then the probability that all three vertices are independent is $\frac{\binom{m}{3}+\binom{m+1}{3}}{\binom{2m+1}{3}}$, with the same result.

2. Show that almost all graphs are not trees.

   For a graph to be a tree it needs to have $(n-1)$ edges for its n vertices, the probability of which is $\binom{\binom{n}{2}}{n-1}/2^{\binom{n}{2}} < (n^2)^n/2^{n^2/4}$. The log of this probability is $2n\ln(n) - (n^2/4)\ln(2)$, which approaches $-\infty$ as $n \to \infty$ since n^2 grows faster than $n\ln(n)$. Therefore, the probability of having $n-1$ edges on n vertices approaches zero as $n \to \infty$.

3. (★) Show that almost all graphs have a triangle.

   Partition the vertices into groups of 3. Then there are at least $n/3$ possible triangles, each of which has a 1/8 chance of being formed at random. The chance that there are no triangles is therefore at most $(7/8)^{n/3}$, which approaches zero very quickly.

   Alternately: by Turan's theorem there must be a triangle if G has at least $n^2/4$ edges, the probability of which approaches 1 as $n \to \infty$.

4. Let $T(G)$ be the number of triangles in a graph G. If G has n vertices then what is $E(T(G))$? (Hard) What is $Var(T(G))$?

   The expected number of triangles is $\binom{n}{3}/8$. Variance is harder.

5. Let $C(G)$ be the number of 4-cycles in a graph G. If G has n vertices then what is $E(C(G))$? (Hard) What is $Var(C(G))$?

   Any four vertices determine two separate 4-cycles, each of which has a 1/16 chance of appearing. The expected number is therefore $(2\binom{n}{4})/16 = \binom{n}{4}/8$.

Turan's Theorem

1. Show that the two formulations of Turan's theorem are equivalent.

   The first statement is the same as saying that $e \le \left(1 - \frac{1}{\omega G}\right)\frac{n^2}{2}$. Let $\overline{e}$ be the number of edges in $\overline{G}$. Then the following statements are therefore all equivalent:

$$\begin{aligned} e &\le \left(1 - \frac{1}{\omega G}\right)\frac{n^2}{2} \\ (n^2 - n)/2 - \overline{e} &\le \left(1 - \frac{1}{\alpha \overline{G}}\right)\frac{n^2}{2} \\ n^2 - n - 2\overline{e} &\le \left(1 - \frac{1}{\alpha \overline{G}}\right) n^2 \\ n - 1 - D &\le \left(1 - \frac{1}{\alpha \overline{G}}\right) n \\ n - (D+1) &\le n - n/\alpha(\overline{G}) \\ n/\alpha(\overline{G}) &\le D + 1 \\ n/(D+1) &\le \alpha \overline{G} \end{aligned}$$

   So the first statement is true for G if and only if the second is true for $\overline{G}$, thus the first is true for *all* graphs if and only if the second is true for all graphs.

2. (★) Define the Turan graph $T(n, r)$ as follows: partition the vertices into r sets of equal or nearly equal size and connect any pair of vertices that are not in the same set. Prove that $T(n, r)$ does not contain K_{r+1} as a subgraph.

   Choose any $(r + 1)$ points: by the Pigeonhole principle we must choose 2 from the same set. These two are not adjacent, therefore our set of $(r + 1)$ points is not a clique.

3. Show that $T(n, r)$ has the maximum number of edges of any n-vertex graph not containing K_{r+1}.

   If $r|n$ then the Turan graph hits the bound given by Turan's Theorem exactly.

   The formal proof in general is somewhat harder, as it turns out. The first step (and the hard part) is to prove that the number of edges are maximized when the vertices are arranged into r sets so that no to vertices in the same set share an edge. If we take this for granted, then to show that $T(n, r)$ has the maximum number of edges we just have to show that the number of edges is maximized when the points are distributed as evenly as possible. To show this: if the graph has sets A and B with $|A| > |B| + 1$, then we can take a point from A and add it to B. The grpah then loses $|B|$ edges but gains $|A| - 1$ edges, for a net gain.

   Your task: Look up and read through the rest of the proof.

Miscellany

1. Define the clique number of a graph, $\omega(G)$, to be the largest m such that K_m is a subgraph of G. Show that $\chi(G) \ge \omega(G)$.

   All vertices in a clique must get a different color since any two of them are adjacent.

2. Show that $\omega(G) = \alpha(\overline{G})$.

   If $S \subset V$ is a clique in G then any two vertices in S are adjacent, and so no two vertices in S are adjacent if $\overline{G}$. Therefore $\alpha(\overline{G}) \ge \omega G$. Similar reasoning in the other direction gives $\omega(G) \ge \alpha \overline{G}$. Therefore the two are equal.