Chapter 10.8: Graph Coloring

Tuesday, August 11

Summary

- Dual graph: formed by putting a vertex at each face of a graph and connecting vertices if the corresponding faces are adjacent.
- Chromatic number ($\chi(G)$): smallest number of colors required to color each vertex of the graph so that no adjacent vertices have the same color.
- Independence number ($\alpha(G)$): the size of the largest set S of vertices such that no two vertices in S share an edge.
- Greedy algorithm: visit the vertices in random order and color each vertex with the first available color.
- If G is planar then $\chi(G) \leq 4$.

Coloring and Independence

1. (★) Find $\alpha(G)$ and $\chi(G)$ for each of the following:
   (a) K_n
   (b) C_n
   (c) P_n
   (d) $K_{m,n}$
2. What is $\chi(G)$, where G is a tree?
3. (★) For any graph G with n vertices, $\chi(G) \leq n - \alpha(G) + 1$.
4. For any graph G with n vertices, $\alpha(G)\chi(G) \geq n$.

More Coloring

1. For every n, find a 2-colorable graph with n vertices such that every vertex has degree $\geq (n-1)/2$.
2. (★) If every vertex in a graph G has degree $\leq d$, then G is $(d+1)$-colorable.
3. (★) If all cycles in a graph G have length divisible by k, then k is k-colorable.
4. Every planar graph is 6-colorable (Hint: induction).
5. If a planar graph has n vertices then $\alpha(G) \geq n/6$.
6. Every triangle-free planar graph has a vertex of degree ≤ 3 (Use a result about the number of edges and vertices from yesterday).
7. Every triangle-free planar graph is 4-colorable.
8. (★) Find a triangle-free planar graph with 11 vertices that is not 3-colorable. (Hint: it has 5-fold rotational symmetry.)

Challenge

1. (★) Find the duals of each of the five Platonic solids.
2. For every graph show that there is some ordering of the vertices for which the greedy algorithm will use $\chi(G)$ colors (the minimum).
3. Find a planar graph and an ordering of the vertices for which the greedy algorithm uses 5 or more colors.

Suggested From Rosen

10.8: 5-10, 13, 32, 35