Chapter 10.8: Graph Coloring

Tuesday, August 11

Summary

- Dual graph: formed by putting a vertex at each face of a graph and connecting vertices if the corresponding faces are adjacent.
- Chromatic number ($\chi(G)$): smallest number of colors required to color each vertex of the graph so that no adjacent vertices have the same color.
- Independence number ($\alpha(G)$): the size of the largest set S of vertices such that no two vertices in S share an edge.
- Greedy algorithm: visit the vertices in random order and color each vertex with the first available color.
- If G is planar then $\chi(G) \leq 4$.

Coloring and Independence

1. ($\bigstar$) Find $\alpha(G)$ and $\chi(G)$ for each of the following:
   (a) K_n: $\alpha(G) = 1$, $\chi(G) = n$
   (b) C_n: $\alpha(G) = \lfloor n/2 \rfloor$, $\chi(G) = 2$ if n is even and 3 if n is odd.
   (c) P_n: $\alpha(G) = \lceil n/2 \rceil$, $\chi(G) = 2$.
   (d) $K_{m,n}$: $\alpha(G) = \max(m, n)$, $\chi(G) = 2$.
2. What is $\chi(G)$, where G is a tree? $\chi(G) = 2$. Prove by induction.
3. ($\bigstar$) For any graph G with n vertices, $\chi(G) \leq n - \alpha(G) + 1$.

   Color $\alpha(G)$ of the vertices with 1 of the colors, then each of the other $n - \alpha(G)$ vertices gets a different color: total number of colors is $n - \alpha(G) + 1$.
4. For any graph G with n vertices, $\alpha(G)\chi(G) \geq n$.

   Let C_i be the set of vertices given color i. Since all of these vertices must be non-adjacent we know that $|C_i| \leq \alpha(G)$ for any i. Therefore for any coloring $n = \sum_i |C_i| \leq \sum_i \alpha(G) \leq \chi(G)\alpha(G)$.

More Coloring

1. For every n, find a 2-colorable graph with n vertices such that every vertex has degree $\geq (n-1)/2$.

   Make a bipartite graph with an equal number of vertices on both sides (or as equal as possible).

2. (★) If every vertex in a graph G has degree $\leq d$, then G is $(d+1)$-colorable.

   Use the greedy algorithm. Any vertex has at most d neighbors so at least 1 color is guaranteed to be available to color that vertex.

3. (★) If all cycles in a graph G have length divisible by $k \geq 2$, then G is k-colorable.

   Rough sketch: Assume that all vertices in the graph have degree ≥ 2. Walk around the graph aimlessly, coloring each vertex you visit with a new color (and starting over at 1 after reaching k). The fact that the length of every cycle is divisible by k means that if you reach the same vertex twice then you will be trying to color it the same color as before, having progressed a multiple of k steps.

   Then color all of the vertices of degree 1.

4. Every planar graph is 6-colorable (Hint: induction).

   Every planar graph has a vertex of degree ≤ 5, so after coloring the rest of the graph this vertex will be guaranteed to have a legal coloring. Remove this vertex, find (by inductive assumption) a 6-coloring of the remaining planar graph, then color the last vertex.

5. If a planar graph has n vertices then $\alpha(G) \geq n/6$.

   Use the previous result along with the result that $\alpha(G)\chi(G) \geq n$.

6. Every triangle-free planar graph has a vertex of degree ≤ 3 (Use a result about the number of edges and vertices from yesterday).

   If a graph with at least 3 vertices is triangle-free, then by a result yesterday $e \leq 2v-4$, so $\sum_v deg(v) = 2e = 4v - 8 < 4v$. Thus there must be some vertex with degree ≤ 3.

7. Every triangle-free planar graph is 4-colorable.

   Induction. Remove a vertex with degree ≤ 3, color the rest of the graph with 4 colors, then color the remaining vertex.

8. (★) Find a triangle-free planar graph with 11 vertices that is not 3-colorable. (Hint: it has 5-fold rotational symmetry.)

   Good luck!

Challenge: Greedy Algorithm

1. For every graph show that there is some ordering of the vertices for which the greedy algorithm will use $\chi(G)$ colors (the minimum).

2. Find a planar graph and an ordering of the vertices for which the greedy algorithm uses 5 or more colors.

Suggested From Rosen

10.8: 5-10, 13, 32, 35