Chapter 11.5: Euler and Hamilton Paths

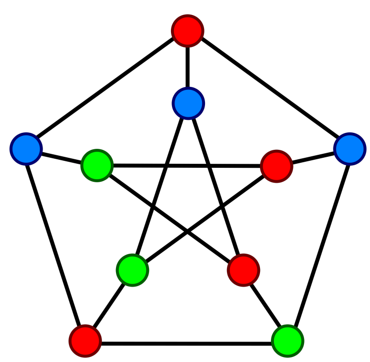
Friday, August 7

Summary

- Euler trail/path: A walk that traverses every edge of a graph once.
- Eulerian circuit: An Euler trail that ends at its starting vertex.
- Eulerian path exists iff graph has ≤ 2 vertices of odd degree.
- Hamilton path: A path that passes through every edge of a graph once.
- Hamilton cycle/circuit: A cycle that is a Hamilton path.
- If G is simple with $n \geq 3$ vertices such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u, v in G, then G has a Hamilton cycle.
- Euler's Formula for plane graphs: $v - e + r = 2$.

Trails and Circuits

1. For which values of n do K_n, C_n, and $K_{m,n}$ have Euler circuits? What about Euler paths? (★)
2. Prove that the dodecahedron is Hamiltonian.
3. A *knight's tour* is a a sequence of legal moves on a board by a knight (moves 2 squares horizontally or vertically, then 1 square at a right angle) that visits each square once. Call it *reentrant* if the tour ends at its starting square.
   (a) Draw graphs that represent the legal moves of the knight on a 3×3 and 3×4 chessboard. (★)
   (b) Show that there is a knight's tour on a 3×4 board but not a 3×3 board.
   (c) Show that if a graph has a Hamilton path then after deleting k vertices, the remaining graph has $\leq k + 1$ connected components.
   (d) Use the previous result to show that there is no knight's tour on a 4×4 chessboard.
4. Show that the Petersen graph does not have a Hamilton circuit, but if you delete any vertex (and all incident edges) then the resulting subgraphs does have a Hamilton circuit.



Grid Graphs

1. (★) Count the number of edges in the $k \times l$ grid.
2. Count the number of shortest paths between opposite corners of a grid.
3. Prove that all grid graphs have a Hamilton path.
4. (★) For what values of k and l does the $k \times l$ grid graph have a Hamilton cycle?
5. You have an 8-by-8 chessboard and 31 dominoes. Is it possible to tile the chessboard with the dominoes if you remove a pair of opposite corners from the board? (Look for an ah-ha! proof.)
6. A mouse wants to eat a $3 \times 3 \times 3$ block of cheese cubes by eating adjacent blocks one after the other. Is it possible for the mouse to eat the center cube last?

Plane Graphs

1. Try to tile the plane with triangles, 5 triangles to a vertex. What happens?
2. Try to tile the plane with triangles, 7 triangles to a vertex. Now what happens?

Suggested From Rosen

10.6: 49, 55, 62, 63