Chapter 11.5: Euler and Hamilton Paths

Friday, August 7

Summary

- Euler trail/path: A walk that traverses every edge of a graph once.
- Eulerian circuit: An Euler trail that ends at its starting vertex.
- Eulerian path exists iff graph has ≤ 2 vertices of odd degree.
- Hamilton path: A path that passes through every edge of a graph once.
- Hamilton cycle/circuit: A cycle that is a Hamilton path.
- If G is simple with $n \geq 3$ vertices such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u, v in G, then G has a Hamilton cycle.
- Euler's Formula for plane graphs: $v - e + r = 2$.

Trails and Circuits

1. For which values of n do K_n, C_n, and $K_{m,n}$ have Euler circuits? What about Euler paths?

   K_n has an Euler circuit for odd numbers $n \geq 3$, and also an Euler path for $n = 2$.

2. (★) Prove that the dodecahedron is Hamiltonian.

   One solution presented in Rosen, p. 699

3. A *knight's tour* is a a sequence of legal moves on a board by a knight (moves 2 squares horizontally or vertically, then 1 square at a right angle) that visits each square once. Call it *reentrant* if the tour ends at its starting square.

   (a) Draw graphs that represent the legal moves of the knight on a 3×3 and 3×4 chessboard.

   (b) (★) Show that there is a knight's tour on a 3×4 board but not a 3×3 board.

   There is no knight's tour on the 3×3 board in particular because there is no way to reach the middle square from the other squares. If we number the squares 1-12 on the 3×4 grid going across each row in succession, then one solution would be 1-8-3-4-11-6-7-2-9-10-5-12. There is, however, no reentrant knight's tour (i.e. a Hamilton cycle). One explanation is that the sequence of moves 6-11-4 (or 4-11-6) must appear in the cycle, but this sequence of moves splits the graph of possible moves in two: 1-8-3 on one side and the rest of the squares on the other. The solution for a Hamilton path started on one side and ended on the other, but a Hamilton cycle does not have this option.

   (c) Show that if a graph has a Hamilton path then after deleting k vertices, the remaining graph has $\leq k + 1$ connected components.

   Verify first for a path: removing a single vertex from a path splits the path into at most 2 connected components. Then the claim that removing k vertices splits a path into $\leq k + 1$ components follows by induction.

   Suppose a graph has a Hamilton path, and order the vertices according to that path. Then use the above result, plus the fact that any additional edges in the graph can only reduce the number of connected components.

   (d) Use the previous result to show that there is no knight's tour on a 4×4 chessboard.

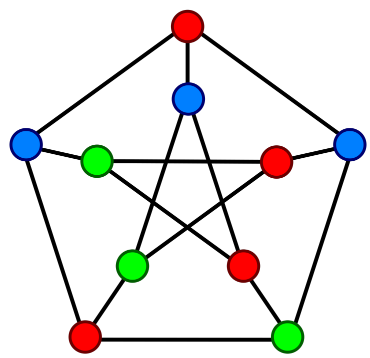
   Number the squares 1-16 going across rows. Removing the four middle squares (6,7,10,11) splits the graph into six connected components.

4. Show that the Petersen graph does not have a Hamilton circuit, but if you delete any vertex (and all incident edges) then the resulting subgraphs does have a Hamilton circuit.

   This basically has to be done by brute force, but we can try to narrow down the number of possibilities that we have to try by using the fact that the Petersen graph is highly symmetrical. In particular, for any two paths of length 3 there is an automorphism that takes one to the other (try to verify this!)

   Call the outside vertices A, B, C, D, E in order and the corresponding inside vertices a, b, c, d, e. Because of the symmetry of the graph we can assume without loss of generality that our cycle contains $A - B - C - D$. It must therefore contain either $D - E$ or $E - A$... assume again without loss of generality that it has $D - E$. Then the vertices b, c, d are constrained so that the cycle must also have the edges $e - b - d$ (for b), $a - c - e$ (for c), and $b - d - a$ (for d). But together these make the five-pointed star $a - c - e - b - d - a$, which makes a Hamilton cycle impossible.

   Again because of symmetry, to show that a cycle exists if any vertex is removed we only need to find a solution for any one particular vertex... let it be C. Then a cycle is $A - B - b - e - c - a - d - D - E - A$.



Grid Graphs

1. (★) Count the number of edges in the $k \times l$ grid.

   Count horizontal and vertical edges separately: the number of edges is $(k-1)l + (l-1)k = 2kl - k - l$.

2. Count the number of shortest paths between opposite corners of a grid.

   It's Pascal's triangle: the number of shortest paths is $\binom{(k-1)+(l-1)}{k-1} = \binom{(k-1)+(l-1)}{l-1}$.

3. Prove that all grid graphs have a Hamilton path.

   Wind around row by row.

4. (★) For what values of k and l does the $k \times l$ grid graph have a Hamilton cycle?

   Count by black-white squares: a Hamilton cycle exists if and only if kl is even.

5. You have an 8-by-8 chessboard and 31 dominoes. Is it possible to tile the chessboard with the dominoes if you remove a pair of opposite corners from the board? (Look for an ah-ha! proof.)

   Nope. If you color the chessboard black and white, then the squares you remove would be the same color, so there are more squares of one color than the other left over on the board.

6. A mouse wants to eat a $3 \times 3 \times 3$ block of cheese cubes by eating adjacent blocks one after the other. Is it possible for the mouse to eat the center cube last?

   No. If you color the cubes on the corners and on the center of each face black and the rest white, you have 14 black cubes and 13 white cubes (including the center one). Any Hamilton path must start and end on a black cube.

Plane Graphs

1. Try to tile the plane with triangles, 5 triangles to a vertex. What happens?

   You will be forced to make the triangles more and more obtuse.

2. Try to tile the plane with triangles, 7 triangles to a vertex. Now what happens?

   You will be forced to make the smallest angles of the triangles smaller and smaller.

Suggested From Rosen

10.6: 49, 55, 62, 63