Chapter 11.1(?): Trees

Thursday, August 6

Summary

- A tree is a connected graph with no cycles.
- The following are equivalent for a graph G with n vertices:
  1. G is a tree
  2. G has $(n-1)$ edges and no cycles
  3. G has $(n-1)$ edges and is connected
- There is a unique path between any 2 vertices in a tree.
- Every tree with at least 2 vertices has at least 2 vertices of degree 1.
- Every tree is bipartite.
- Removing any edge from a tree will separate the tree into 2 connected components.

Molecules and Friends

1. (★) Show that $C_nH_{2n+1}OH$ has a tree structure (carbon makes 4 bonds, hydrogen 1, and oxygen 2).
2. (★) Show that C_6H_6 does not have a tree structure.
3. (★) Find all isomers (non-isomorphic graphs) of pentane (C_5H_{12}).
4. Find all isomers of hexane (C_6H_{14}).
5. (★) How many edges does a tree with 10,000 vertices have?
6. How many trees (up to isomorphism) are there with 2 vertices of degree 3, 1 vertex of degree 2, and all the other vertices have degree 1? Draw them.
7. (★) Which complete bipartite graphs $K_{m,n}$ are trees?

Proofs

1. Prove that removing any edge from a tree will result in a graph with 2 connected components.
2. Prove: There is a unique path between any two vertices in a tree (assume there are two distinct paths, then use that to find a cycle–contradiction. It may help to draw a picture).
3. (★) Prove: A tree with at least 2 vertices has at least 2 vertices of degree 1 (find the longest path in the tree, look at the endpoints).
4. (★) Show that every tree is bipartite.
5. (Harder) Let l be the length of the longest path in a tree. Prove: any 2 paths of length l have a common vertex (assume that there are 2 that do not, then find a contradiction).

Challenge Puzzle

1. 8 knights enter a single-round elimination tournament with balanced brackets (quarterfinals, semifinals, finals). All knights are evenly matched and have a 1/2 chance of winning any given bout. If two of the knights are twins, what is the probability that they fight each other at some point during the tournament?

Suggested From Rosen

11.2: 6-7 (puzzles) 11.4: 1-6 Supplement (p. 807): 35-37