

Chapter 11.1(?): Trees

Thursday, August 6

Summary

- A tree is a connected graph with no cycles.
- The following are equivalent for a graph G with n vertices:
 1. G is a tree
 2. G has $(n - 1)$ edges and no cycles
 3. G has $(n - 1)$ edges and is connected
- There is a unique path between any 2 vertices in a tree.
- Every tree with at least 2 vertices has at least 2 vertices of degree 1.
- Every tree is bipartite.
- Removing any edge from a tree will separate the tree into 2 connected components.

Molecules and Friends

1. (★) Show that $C_nH_{2n+1}OH$ has a tree structure (carbon makes 4 bonds, hydrogen 1, and oxygen 2).
The molecule is connected with $3n + 3$ vertices and $\frac{1}{2}(4n + 2n + 1 + 2 + 1) = 3n + 2$ edges, and is thus a tree.
2. (★) Show that C_6H_6 does not have a tree structure.
The molecule has 12 vertices and $\frac{1}{2}(24 + 6) = 15$ edges, so is not a tree.
3. (★) Find all isomers (non-isomorphic graphs) of pentane (C_5H_{12}).
There are three of them.
4. Find all isomers of hexane (C_6H_{14}).
There are five of them.
5. (★) How many edges does a tree with 10,000 vertices have?
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6. How many trees (up to isomorphism) are there with 2 vertices of degree 3, 1 vertex of degree 2, and all the other vertices have degree 1? Draw them.
We just have to decide how to arrange the vertices with degree numbers more than 1 relative to each other, and the degree 1 vertices will be placed automatically (like hydrogen around a carbon chain). The two options are 2-3-2 and 2-2-3 (which is isomorphic to 3-2-2).
7. (★) Which complete bipartite graphs $K_{m,n}$ are trees?
Precisely those where $n = 1$ or $m = 1$. As proof, we need $mn = m + n - 1$. If $m, n \geq 2$ then $mn \geq 2 \max(m, n) \geq m + n > m + n - 1$, a contradiction.

Proofs

1. Prove that removing any edge from a tree will result in a graph with 2 connected components.

Let (u,v) be the edge removed. Since there is a unique path between any 2 vertices in a tree and a path between u and v has been removed, u and v are no longer connected.

To show that the graph has 2 connected components, we need to show that every other vertex is connected to either u or v . Suppose that w is no longer connected to u . This means that the unique path from w to u had (u,v) as an edge, and therefore that the path passed through v and ended in the edge (u,v) . Thus w and v are connected.

2. Prove: There is a unique path between any two vertices in a tree (assume there are two distinct paths, then use that to find a cycle—contradiction. It may help to draw a picture).

Suppose there are two paths from u to v : $(u, x_1, x_2, \dots, x_n, v)$ and $(u, y_1, y_2, \dots, y_n, v)$. These paths both begin with the vertex u , so are the same up to some point... call the last point where they are the same i . Then find the next points x_j, y_k such that $j, k > i$ and $x_j = y_k$.

Then take these two sections of the paths (x_i, \dots, x_j) and (y_i, \dots, y_k) . Joining them together will produce a cycle, which is a contradiction. Therefore no two such paths exist.

3. (★) Prove: A tree with at least 2 vertices has at least 2 vertices of degree 1 (find the longest path in the tree, look at the endpoints).

Find the longest path, and suppose that one endpoint u has degree ≥ 2 . If any of its other neighbors are part of the longest path, this would produce a cycle. If its other neighbors are not part of the longest path, we can find a longer path. Either of these is a contradiction, so the endpoints must have degree 1.

4. (★) Show that every tree is bipartite.

One method is to use induction: A tree with 1 or 2 vertices is bipartite.

For the inductive step, remove all of the vertices of degree 1. A smaller tree remains, which by the inductive hypothesis can be colored with 2 colors. Then put all the degree 1 vertices back in, and color each of them the opposite of their neighbor's color. This completes the proof.

5. (Harder) Let l be the length of the longest path in a tree. Prove: any 2 paths of length l have a common vertex (assume that there are 2 that do not, then find a contradiction).

Suppose there are two paths with no vertex in common. There is some path that joins these two paths, say from u to v . The point u occurs somewhere along path 1 and the point v along path 2. Consider the two parts of the paths split by u and v , and pick the longer of the two. These two parts combined with the path joining u and v is of length $\geq l/2 + l/2 + 1 > l$, contradicting the assumption that l was the length of the longest path. Therefore any two such paths share a vertex.

Challenge Puzzle

1. 8 knights enter a single-round elimination tournament with balanced brackets (quarterfinals, semifinals, finals). All knights are evenly matched and have a $1/2$ chance of winning any given bout. If two of the knights are twins, what is the probability that they fight each other at some point during the tournament?

Suggested From Rosen

11.2: 6-7 (puzzles) 11.4: 1-6 Supplement (p. 807): 35-37