Chapter 10.2-10.3: Graph Isomorphism and Connectedness

Wednesday, August 5

Summary

- An isomorphism from $G = (V, E)$ to $H = (W, F)$ is a bijection $\varphi : V \to W$ such that $(u, v) \in E$ if and only if $(\varphi(u), \varphi(v)) \in F$.
- An automorphism is an isomorphism from G to itself.
- A walk is a sequence of edges such that successive edges share a common vertex.
- A path is a walk with no repeated vertices.
- A cycle is a path that ends where it began.
- A trail is a walk with no repeated edges.
- A graph is connected if any two vertices are connected by a path.
- The distance between two vertices is the length of the shortest path connecting them.
- The diameter of a graph is the maximum distance between two vertices.

Isomorphisms

1. (★) Show that C_5 and $\overline{C_5}$ are isomorphic.

   $\overline{C_5}$ is a five-pointed star, which is the same as a cycle of length 5 under the appropriate isomorphism.

2. (★) Show that C_4 and $\overline{C_4}$ are not isomorphic.

   $\overline{C_4}$ has only 2 edges but C_4 has 4.

3. Show that if G has n vertices and G is isomorphic to $\overline{G}$ then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ (Hint: count edges).

   If they are isomorphic then G and $\overline{G}$ must have the same number of edges, thus the number of possible edges $\binom{n}{2}$ is even. So $2|n(n-1)/2$, and therefore $4|n(n-1)$. Since n and $n-1$ are relatively prime, this can only happen if $4|n$ or $4|(n-1)$.

4. (★) Find a graph G on 4 vertices such that G and $\overline{G}$ are isomorphic.

   The path of length 3 (P_3) is isomorphic to its complement.

5. (★) How many automorphisms are there on C_n? K_n?

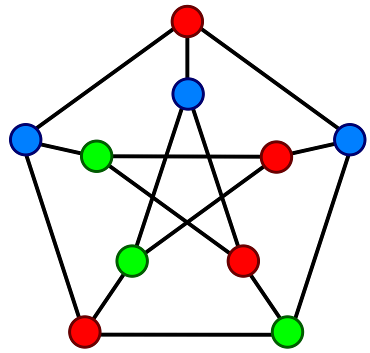
   C_n has $2n$ automorphisms and K_n has $n!$.

6. Find all nonisomorphic graphs with 4 vertices.

   There are 11 of them: 1 with 0 edges, 1 with 1, 2 with 2, 3 with 3, 2 with 4, 1 with 5, and 1 with 6.

7. (Challenge) Show that the Petersen graph, shown below, has 120 automorphisms.

   Good luck!



Paths and Connectedness

1. (★) Find the diameter of $P_n, C_n, K_n, K_{m,n}$.

   The diameter of P_n is $n-1$, that of C_n is $\lfloor n/2 \rfloor$, that of K_n is 1, and that of $K_{m,n}$ is 2.

2. If v has odd degree in G then there is some w of odd degree such that v and w are connected by a path.

   The Handshake Theorem (the number of vertices with odd degree is even) holds for the connected components of a graph as well as the entire graph. Thus if there is at least one vertex of odd degree in a component C then there must be another as well. Since C is connected, the two vertices are connected by a path.

3. Find all non-isomorphic trees with 6 vertices.

   There are 6 of them.

4. Find a graph with n vertices, $n-1$ edges, and diameter 2.

   Make a wheel with one vertex in the center connected to each of the $n-1$ vertices around the edges.

5. (★) Count the number of 4-cycles in $K_{m,n}$.

   Choose 2 vertices on one side and 2 on the other; any such choice of 4 vertices determines a cycle and any 4-cycle must have 2 vertices on each side. The number of 4-cycles is therefore $\binom{m}{2}\binom{n}{2}$.

6. Prove that $d(x,y) + d(y,z) \geq d(x,z)$ for any vertices $x, y, z \in G$. Find an example of a graph and 3 vertices in the graph where the two sides are not equal.

   Just append the shortest path from x to y to the path from y to z to get a walk (not necesarily a path!) from x to z. The length of the shortest path must be shorter than the length of this walk.

   Pick the 3 vertices of K_3: $d(x,y) = d(y,z) = d(x,z) = 1$, so $d(x,y) + d(y,z) > d(x,z)$.

7. Prove: If every vertex in a graph G has degree at least 2 then then G contains a cycle.

   Start at any vertex, and begin walking. Since each vertex has degree at least 2 we will never reach a dead end. Since the number of vertices in G is finite, we will eventually come to the same vertex twice. The part of the path that starts and ends at this vertex makes a cycle.

Suggested From Rosen

10.3: 34-44, 45-46, 53-56, 66
10.4: 19-25, 45, 64