Summary

- An isomorphism from $G = (V,E)$ to $H = (W,F)$ is a bijection $\varphi : V \rightarrow W$ such that $(u,v) \in E$ if and only if $(\varphi(u),\varphi(v)) \in F$.
- An automorphism is an isomorphism from $G$ to itself.
- A walk is a sequence of edges such that successive edges share a common vertex.
- A path is a walk with no repeated vertices.
- A cycle is a path that ends where it began.
- A trail is a walk with no repeated edges.
- A graph is connected if any two vertices are connected by a path.
- The distance between two vertices is the length of the shortest path connecting them.
- The diameter of a graph is the maximum distance between two vertices.

Isomorphisms

1. (★) Show that $C_5$ and $\overline{C_5}$ are isomorphic.
   $\overline{C_5}$ is a five-pointed star, which is the same as a cycle of length 5 under the appropriate isomorphism.

2. (★) Show that $C_4$ and $\overline{C_4}$ are not isomorphic.
   $\overline{C_4}$ has only 2 edges but $C_4$ has 4.

3. Show that if $G$ has $n$ vertices and $G$ is isomorphic to $\overline{G}$ then $n \equiv 0 \pmod{4}$ or $n \equiv 1 \pmod{4}$ (Hint: count edges).
   If they are isomorphic then $G$ and $\overline{G}$ must have the same number of edges, thus the number of possible edges $\binom{n}{2}$ is even. So $2|n(n-1)/2$, and therefore $4|n(n-1)$. Since $n$ and $n-1$ are relatively prime, this can only happen if $4|n$ or $4|(n-1)$.

4. (★) Find a graph $G$ on 4 vertices such that $G$ and $\overline{G}$ are isomorphic.
   The path of length 3 ($P_3$) is isomorphic to its complement.

5. (★) How many automorphisms are there on $C_n$? $K_n$?
   $C_n$ has $2n$ automorphisms and $K_n$ has $n!$.

6. Find all nonisomorphic graphs with 4 vertices.
   There are 11 of them: 1 with 0 edges, 1 with 1, 2 with 2, 3 with 3, 2 with 4, 1 with 5, and 1 with 6.

7. (Challenge) Show that the Petersen graph, shown below, has 120 automorphisms.
   Good luck!
Paths and Connectedness

1. (⋆) Find the diameter of $P_n$, $C_n$, $K_n$, $K_{m,n}$.
   
   The diameter of $P_n$ is $n - 1$, that of $C_n$ is $\lceil n/2 \rceil$, that of $K_n$ is 1, and that of $K_{m,n}$ is 2.

2. If $v$ has odd degree in $G$ then there is some $w$ of odd degree such that $v$ and $w$ are connected by a path.
   
   The Handshake Theorem (the number of vertices with odd degree is even) holds for the connected components of a graph as well as the entire graph. Thus if there is at least one vertex of odd degree in a component $C$ then there must be another as well. Since $C$ is connected, the two vertices are connected by a path.

3. Find all non-isomorphic trees with 6 vertices.
   
   There are 6 of them.

4. Find a graph with $n$ vertices, $n - 1$ edges, and diameter 2.
   
   Make a wheel with one vertex in the center connected to each of the $n - 1$ vertices around the edges.

5. (⋆) Count the number of 4-cycles in $K_{m,n}$.
   
   Choose 2 vertices on one side and 2 on the other; any such choice of 4 vertices determines a cycle and any 4-cycle must have 2 vertices on each side. The number of 4-cycles is therefore $\binom{m}{2} \binom{n}{2}$.

6. Prove that $d(x, y) + d(y, z) \geq d(x, z)$ for any vertices $x, y, z \in G$. Find an example of a graph and 3 vertices in the graph where the two sides are not equal.
   
   Just append the shortest path from $x$ to $y$ to the path from $y$ to $z$ to get a walk (not necesarily a path) from $x$ to $z$. The length of the shortest path must be shorter than the length of this walk.
   
   Pick the 3 vertices of $K_3$: $d(x, y) = d(y, z) = d(x, z) = 1$, so $d(x, y) + d(y, z) > d(x, z)$.

7. Prove: If every vertex in a graph $G$ has degree at least 2 then then $G$ contains a cycle.

   Start at any vertex, and begin walking. Since each vertex has degree at least 2 we will never reach a dead end. Since the number of vertices in $G$ is finite, we will eventually come to the same vertex twice. The part of the path that starts and ends at this vertex makes a cycle.

Suggested From Rosen

10.3: 34-44, 45-46, 53-56, 66
10.4: 19-25, 45, 64