Chapter 7.4: Expected Value and Variance

Monday, August 3

Summary

- Covariance of X and Y is $E([X - E(X)][Y - E(Y)]) = E(XY) - E(X)E(Y)$
- Correlation of X and Y is $\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$.
- $G = (V, E)$. V = set of vertices, E = $\{(u, v), \ldots\}$ = set of edges.
- Digraph: the direction of the edge matters, (u,v) = arc from u to v.
- *degree* of u = number of neighbors, *isolated* = degree 0
- K_n = complete graph on n vertices, C_n = cycle
- *bipartite* if it can be colored with 2 colors so that neighbors have different colors, or if we can write $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ so that every edge in E has one elt. in V_1 and one in V_2.
- Induced subgraph: choose the vertices; the edges are automatic.
- Complement: $\overline{G}$ has all of the edges that G doesn't.

Covariance/Correlation

1. Flip 3 coins. Let X be the number of heads flipped and let Y be the number of times the sequence HH appears. Find $Cov(X, Y)$.
2. (★) Let X be the number of heads in 2 flips of a fair coin, and let Y be the number of tails. Show that the correlation of X and Y is -1.
3. Show that if $X + Y$ is any constant then the correlation of X and Y is -1.
4. Show that the correlation of X and Y is equal to the correlation of $(aX + b)$ and Y.
5. Prove that $Cov(X, Y)^2 \leq Var(X)Var(Y)$ for any random variables X and Y (This is equivalent to the Cauchy-Schwarz inequality. Hint: Find the a that minimizes $Var(X - aY)$ and use the fact that the variance must be at least 0.)

Graphs

1. If $G = (V, E)$ is a graph, show that $|E| \leq \binom{n}{2}$.
2. ($\bigstar$) Show that $|E(G)| + |E(\overline{G})| = \binom{n}{2}$.
3. The number of graphs with n (labeled) vertices is $2^{\binom{n}{2}}$
4. ($\bigstar$) Let G have n vertices and m edges. How many induced subgraphs are there? How many spanning subgraphs are there?
5. How many spanning subgraphs of K_n are there with exactly m edges?
6. Show that if G is bipartite then $|E| \leq \lfloor n^2/4 \rfloor$
7. ($\bigstar$) How many edges does K_n have? What about C_n? $K_{m,n}$?
8. For which values of n is K_n bipartite? What about C_n?
9. Describe and count the edges of $\overline{K_n}, \overline{C_n}, \overline{K_{m,n}}$.
10. ($\bigstar$) Draw a directed graph on the 7 vertices $\{0, 1, \ldots, 6\}$ where (u, v) is an edge if and only if $v \equiv 3u \pmod 7$.
11. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree (Hint: pigeon-hole principle)
12. ($\bigstar$) How many triangles does the graph K_n contain?
13. (Hard) If G is triangle-free, then $|E(G)| \leq \lfloor n^2/4 \rfloor$ (use inductions in increments of 2, delete both vertices connected by some edge for the inductive step).

Suggested From Rosen

10.2: 1-5, 7, 19, 21-25, 53-55