

Chapter 7.4: Expected Value and Variance

Wednesday, July 28

Summary

- Expected Value: $E(X) = \sum_{s \in S} p(s)X(s) = \sum_{r \in X(s)} r \cdot p(X = r)$
- Expected value of a Bernoulli trial with probability p is p .
- $E(X + Y) = E(X) + E(Y)$ for any random variables X and Y .
- $E(aX + b) = a \cdot E(X) + b$
- X and Y are independent: $(\forall r, s \in \mathbb{R})(p(X = r, Y = s) = p(X = r)p(Y = s))$
- If X and Y are independent then $E(XY) = E(X)E(Y)$.
- Variance: $Var(X) = E([X - E(X)]^2) = E(X^2) - E(X)^2$
- If X and Y are independent then $Var(X + Y) = Var(X) + Var(Y)$
- $Var(aX + b) = a^2 \cdot Var(X)$.

Expected Value

1. If I roll three dice, what is the expected value of the product of the three numbers rolled?
2. If I roll a die and cube the result, what is the expected value?
3. A game of roulette has 25 black numbers, 25 red numbers, and 2 green numbers. You bet even money on a color (black or red). If you bet a dollar, what are your expected winnings?
4. (★) I roll a red die and a blue die and subtract the red number from the blue number. What is the expected value of the result?
5. (★) Roll two dice and take the *positive* difference of the two numbers. What is the expected value of the result?
6. Three friends weigh an average of 135 pounds. What is the most the heaviest friend can weigh?
7. (★) Give an intuitive argument that the average of $\{1, 2, 3, \dots, n\}$ is $(1 + n)/2$ (think scales and symmetry). Conclude (yet again!) that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.
8. What is the average of all numbers between 1 and 100 that are divisible by 7? (Do not add all the numbers.)
9. (★) Some standardized tests have multiple choice questions with 5 options. You get 1 point for a correct answer, -0.25 points for a wrong answer, and 0 points for no answer. If you guess randomly on every question for a 100-question test, what is your expected score?
10. Your professor gives you 10 questions to study for a final exam. You study 8 of them at random and your professor picks 5 at random (independent of your choice) to put on the final. What is the expected number of questions on the final that you will have prepared for?

Indicator Variables

1. 2000 people go to a party and each person brings a hat, checking it on the way in. On their way out the hat checker hands back the hats randomly. What is the expected number of hats that find their correct owner?
2. At the party the host has cards numbered 1 to 2000, and gives one card to each guest. What is the expected number of guests whose card matches the year they were born? (All of the guests can legally drink vodka.)
3. (★) You flip 20 coins. What is the expected number of times you will see the sequence HHH? (THHHHT counts as two HHH sequences.)
4. You roll a die 10 times. What is the expected number of times a pair of consecutive numbers will add up to 8?

Martingales and Friends

1. You have a foolproof strategy for winning money at the casino: You bet a dollar on a game (say, craps) that you have p chance of winning ($0 < p < 0.5$). If you win, great! If you lose, bet two dollars on the next round, then four, then 8, 16, and so on until you win. (Then you start over at 1 dollar). If you play until you win a game, what are your expected winnings?
2. What is the probability that you will eventually win?
3. What are some potential problems with adopting this strategy in real life?
4. Flip a coin until you get tails, and collect 2^n dollars for flipping n heads. How much should you be willing to play this game?