Chapter 7.4: Expected Value and Variance

Wednesday, July 28

Summary

- Expected Value: $E(X) = \sum_{s \in S} p(s)X(s) = \sum_{r \in X(s)} r \cdot p(X = r)$
- Expected value of a Bernoulli trial with probability p is p.
- $E(X+Y) = E(X) + E(Y)$ for any random variables X and Y.
- $E(aX + b) = a \cdot E(X) + b$
- X and Y are independent: $(\forall r, s \in \mathbb{R})(p(X = r, Y = s) = p(X = r)p(Y = s))$
- If X and Y are independent then $E(XY) = E(X)E(Y)$.
- Variance: $Var(X) = E([X - E(X)]^2) = E(X^2) - E(X)^2$
- If X and Y are independent then $Var(X + Y) = Var(X) + Var(Y)$
- $Var(aX + b) = a^2 \cdot Var(X)$.

Expected Value

1. If I roll three dice, what is the expected value of the product of the three numbers rolled?

   The three die rolls are independent, so $E(ABC) = E(A)E(B)E(C) = 3.5^3 = 42.875$.

2. If I roll a die and cube the result, what is the expected value?

   The expected value is $\frac{1}{6}(1 + 8 + 27 + 64 + 125 + 216) = 73.5$.

3. A game of roulette has 25 black numbers, 25 red numbers, and 2 green numbers. You bet even money on a color (black or red). If you bet a dollar, what are your expected winnings?

   The expected winnings are $25/52 - 27/52 = -2/52 = -1/26$. You will (likely) lose money in the long run.

4. (★) I roll a red die and a blue die and subtract the red number from the blue number. What is the expected value of the result?

   $E(B - R) = E(B) - E(R) = 0$.

   Also zero by symmetry, since it shouldn't matter whether you subtract red from blue or blue from red. So $E(B - R) = E(R - B) = -E(B - R)$, so $E(B - R) = 0$.

5. (★) Roll two dice and take the *positive* difference of the two numbers. What is the expected value of the result?

   Basically just make a table. There are 10 ways to get a difference of 1, 8 for 2, 6 for 3, 4 for 4, and 2 for 5, so the expected value is $\frac{1}{36}(10 \cdot 1 + 8 \cdot 2 + 6 \cdot 3 + 4 \cdot 4 + 2 \cdot 5) = \frac{70}{36} = \frac{35}{18}$.

6. Three friends weigh an average of 135 pounds. What is the most the heaviest friend can weigh?

   405 pounds, assuming the other two weigh nothing.

7. (★) Give an intuitive argument that the average of $\{1, 2, 3, \ldots, n\}$ is $(1 + n)/2$ (think scales and symmetry). Conclude (yet again!) that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

   The numbers are spaced symmetrically around $(1 + n)/2$, so that should be the average. The sum is n times the average, and so is equal $n(n + 1)/2$.

8. What is the average of all numbers between 1 and 100 that are divisible by 7? (Do not add all the numbers.)

   Similar logic as in the previous problem, the average is $(7+98)/2 = 105/2$.

9. (★) Some standardized tests have multiple choice questions with 5 options. You get 1 point for a correct answer, -0.25 points for a wrong answer, and 0 points for no answer. If you gess randomly on every question for a 100-question test, what is your expected score?

   $E(G) = (4/5)(-1/4) + (1/5)1 = 0$. You lose nothing (on average) by guessing.

10. Your professor gives you 10 questions to study for a final exam. You study 8 of them at random and your professor picks 5 at random (independent of your choice) to put on the final. What is the expected number of questions on the final that you will have prepared for?

    The expected number of questions that you will have prepared for is

$$\frac{\binom{8}{3}\binom{2}{2}\cdot 3 + \binom{8}{4}\binom{2}{1}\cdot 4 + \binom{8}{5}\binom{2}{0}\cdot 5}{\binom{10}{5}} = \frac{56\cdot 3 + 70\cdot 4 + 56\cdot 5}{252} = 4$$

    Alternately, note that the probability of getting 3 questions right is the same as the probability of getting 5 questions right, so these two scenarios together have an average of getting 4 right. The only other option is to get exactly 4 questions right, and so the average must be 4.

Indicator Variables

1. 2000 people go to a party and each person brings a hat, checking it on the way in. On their way out the hat checker hands back the hats randomly. What is the expected number of hats that find their correct owner?

2. At the party the host has cards numbered 1 to 2000, and gives one card to each guest. What is the expected number of guests whose card matches the year they were born? (All of the guests can legally drink vodka.)

3. (★) You flip 20 coins. What is the expected number of times you will see the sequence HHH? (THHHHT counts as two HHH sequences.)

4. You roll a die 10 times. What is the expected number of times a pair of consecutive numbers will add up to 8?

Martingales and Friends

1. You have a foolproof strategy for winning money at the casino: You bet a dollar on a game (say, craps) that you have p chance of winning ($0 < p < 0.5$). If you win, great! If you lose, bet two dollars on the next round, then four, then 8, 16, and so on until you win. (Then you start over at 1 dollar). If you play until you win a game, what are your expected winnings?

   The chances of eventually winning a round are 1 (i.e. the chances of losing infinitely many times in a row are 0 as long as $p > 0$). In every case the amount you win from the last round makes up for all the money you lost before and leaves you with a gain of 1 dollar (e.g. -1 - 2 - 4 + 8 = 1). So your expected winnings are 1 dollar. (You can also show this by adding up the power series).

2. What is the probability that you will eventually win?

   1

3. What are some potential problems with adopting this strategy in real life?

   First, you may run out of money and have to stop. Second, casinos often have limits on the amount of money you may bet on a single round of a game.

4. Flip a coin until you get tails, and collect 2^n dollars for flipping n heads. How much should you be willing to play this game?

$$
\begin{aligned}
E(G) &= G(T)p(T) + G(HT)p(HT) + G(HHT)p(HHT) + \dots \\
&= 1 \cdot 1/2 + 2 \cdot (1/4) + 4 \cdot (1/8) + \dots \\
&= 1/2 + 1/2 + 1/2 + \dots \\
&= \infty?
\end{aligned}
$$

If you had infinite time and money, you should be willing to pay any amount of money to play this game. In the real world, this might not be a good idea.