

Chapter 7.3: Bayes' Theorem

Tuesday, July 28

Summary

- Conditional Probability: $p(E|F) = \frac{p(E \cap F)}{p(F)}$
- Backwards: $p(E \cap F) = p(E|F)p(F)$
- Bayes' Theorem: $p(F|E) = \frac{p(E|F)p(F)}{p(E)}$
- Restated: $p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$
- Generalized: if F_1, \dots, F_n are disjoint events such that $\bigcup_{i=1}^n F_i = S$, then

$$p(F|E) = \frac{p(E|F)p(F)}{\sum_{i=1}^n p(E|F_i)p(F_i)}$$

Bayes' Theorem

1. (★) Urn A has three red and five black balls and Urn B has two red and seven black. You pick an urn at random and draw a red ball from it. What is the probability that it was Urn A?
2. (★) Suppose there was a 2/3 chance (rather than 1/2) to begin with that you picked Urn B. Now what is your estimate that you have picked Urn A?
3. (★) Now there is also Urn C, which has six red balls and one black. You have a 40% chance of picking Urn A, a 30% chance of picking Urn B, and a 30% chance of picking Urn C. You select an urn and draw a black ball from it. What are the probabilities that it was Urn A? Urn B? Urn C?

Compounding Information

1. (★) You are a professional coin flipper looking for biased coins at the mint. You know that 99 out of every 100 coins are perfectly fair and that 1 out of 100 lands on heads 60% of the time. You flip a coin 50 times and get 33 heads. What are the odds that this coin is biased?
2. You decided that this coin is suspicious and flip it 50 more times, getting another 29 heads. If your priors are taken from the answer to the previous question, what are the new odds that the coin is biased?
3. Suppose instead that you just flipped the coin 100 times to begin with and got 33 heads in the first 50 flips and 29 heads in the next 50. Now what are the odds that the coin is biased? [EDIT: The problem as originally worded would not lead to the same answer as in the previous question.]
4. Prove that these two answers will in general be the same.

Without Loss of Generality...

1. You and a friend both pick 5 random integers between 1 and 10. What is the probability that you have exactly 3 in common?
2. Flip a coin until the sequence HHHTH ever appears (win) or the sequence TTTHT ever appears (lose). What is your chance of winning?
3. Flip a coin many times. You win if the sequence HHH ever appears and lose if the sequence THH ever appears. What is your chance of winning? (Hint: there is overlap between the two sequences!)
4. Flip a coin until you either have 1 more tail than heads (lose) or have 3 more heads than tails (win). What is your chance of winning?
5. An urn has 837 red balls, 3 blue balls, and 2 green balls. You take balls out one at a time until you draw a blue (win) or a green (lose). What is your chance of winning?