Chapter 7.3: Bayes' Theorem

Tuesday, July 28

Summary

- Conditional Probability: $p(E|F) = \dfrac{p(E \cap F)}{p(F)}$
- Backwards: $p(E \cap F) = p(E|F)p(F)$
- Bayes' Theorem: $p(F|E) = \dfrac{p(E|F)p(F)}{p(E)}$
- Restated: $p(F|E) = \dfrac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$
- Generalized: if $F_1, \ldots, F_n$ are disjoint events such that $\bigcup_{i=1}^{n} F_i = S$, then

$$p(F|E) = \frac{p(E|F)p(F)}{\sum_{i=1}^{n} p(E|F_i)p(F_i)}$$

Bayes' Theorem

1. Urn A has three red and five black balls and Urn B has two red and seven black. You pick an urn at random and draw a red ball from it. What is the probability that it was Urn A?

   $p(A|r) = \frac{p(r|A)p(A)}{p(r|A)p(A)+p(r|B)p_B} = \frac{3/8}{3/8+2/9} = .63.$

2. Suppose there was a 2/3 chance (rather than 1/2) to begin with that you picked Urn B. Now what is your estimate that you have picked Urn A?

   $p(A|r) = \frac{p(r|A)p(A)}{p(r|A)p(A)+p(r|B)p_B} = \frac{(3/8)(1/3)}{(3/8)(1/3)+(2/9)(2/3)} = .46$. Bigger than 1/3, but less than the estimate for the previous problem.

3. Now there is also Urn C, which has six red balls and one black. You have a 40% chance of picking Urn A, a 30% chance of picking Urn B, and a 30% chance of picking Urn C. You select an urn and draw a black ball from it. What are the probabilities that it was Urn A? Urn B? Urn C?

   $p(k|A)p(A) = (5/8)(2/5) = 1/4 = 105/420$, $p(k|B)p(B) = (7/9)(3/10) = 7/30 = 98/420$, $p(k|C)p(C) = (1/7)(3/10) = 3/70 = 18/420$, and the sum of all 3 is 221/420. So the chance that you have Urn A is 105/221, the chance of Urn B is 98/221, and the chance of Urn C is 18/221.

Compounding Information

1. You are a professional coin flipper looking for biased coins at the mint. You know that 99 out of every 100 coins are perfectly fair and that 1 out of 100 lands on heads 60% of the time. You flip a coin 50 times and get 33 heads. What are the odds that this coin is biased?

   Let B = biased, F = the chance of the flip. Then

$$
\begin{aligned}
p(B|F) &= \frac{p(F|B)p(B)}{p(F|B)p(B) + p(F|\overline{B})p(\overline{B})} \\
&= \frac{\binom{50}{33}(0.6)^{33}(0.4)^{17}(1/100)}{\binom{50}{33}(0.6)^{33}(0.4)^{17}(1/100) + \binom{50}{33}(0.5)^{50}(99/100)} \\
&= \frac{(0.6)^{33}(0.4)^{17}(1/100)}{(0.6)^{33}(0.4)^{17}(1/100) + (0.5)^{50}(99/100)} \\
&= \frac{1.2^{33}0.8^{17}(0.01)}{0.99 + 1.2^{33}0.8^{17}(0.01)} && \approx \frac{.09237}{.99 + .09237} \\
&\approx 0.0853
\end{aligned}
$$

   There is about an 8.5% chance that the coin is biased.

2. You decided that this coin is suspicious and flip it 50 more times, getting another 29 heads. If your priors are taken from the answer to the previous question, what are the new odds that the coin is biased?

$$
\begin{aligned}
p(B|F) &= \frac{p(F|B)p(B)}{p(F|B)p(B) + p(F|\overline{B})p(\overline{B})} \\
&= \frac{1.2^{33}0.8^{17}(0.0853)}{0.915 + 1.2^{33}0.8^{17}(0.0853)} \\
&\approx \frac{.788}{.915 + .788} \\
&\approx .463
\end{aligned}
$$

   The chance that the coin is biased is now a little less than 50%.

3. Suppose instead that you just flipped the coin 100 times to begin with and got 33 heads in the first 50 flips and 29 heads in the next 50. Now what are the odds that the coin is biased? [EDIT: The problem as originally worded would not lead to the same answer as in the previous question.]

   Answer will be the same. . . see the result below.

4. Prove that these two answers will in general be the same.

   If F_1 is the result from the first flip and F_2 is the result from the second flip, then we would like to show that the result for $p(B|F_1 \cap F_2)$ will be the same whether we update our expectations for the two flips at the same time or for one after the other. By Bayes' Theorem, this comes down to showing

$$
\frac{p(F_1 \cap F_2|B)p(B)}{p(F_1 \cap F_2)} = \frac{p(F_2|B)p(B|F_1)}{p(F_2|F_1)}
$$

This is true since

$$\begin{aligned}
\frac{p(F_1 \cap F_2|B)p(B)}{p(F_1 \cap F_2)} &= p(B|F_1 \cap F_2) \\
&= \frac{p(B \cap F_1 \cap F_2)}{p(F_1 \cap F_2)} \\
&= \frac{p(F_2|B \cap F_1)p(B \cap F_1)}{p(F_1 \cap F_2)} \\
&= \frac{p(F_2|B \cap F_1)p(B|F_1)p(F_1)}{p(F_2|F_1)p(F_1)} \\
&= \frac{p(F_2|B)p(B|F_1)}{p(F_2|F_1)}
\end{aligned}$$

Note that $p(F_2|B \cap F_1) = p(F_2|B)$ because once the coin is fixed (at B) the outcomes of the flips F_1 and F_2 are independent. Interestingly, F_1 and F_2 are not independent events in general.

Without Loss of Generality...

1. You and a friend both pick 5 random integers between 1 and 10. What is the probability that you have exactly 3 in common?

   WLOG you can pick your numbers to be 1,2,3,4,5. Then the chance that you have exactly 3 in common is $\binom{5}{3}\binom{5}{2}/\binom{10}{5} = 100/252 \approx .397$

2. Flip a coin until the sequence HHHTH ever appears (win) or the sequence TTTHT ever appears (lose). What is your chance of winning?

   1/2, by symmetry.

3. Flip a coin many times. You win if the sequence HHH ever appears and lose if the sequence THH ever appears. What is your chance of winning? (Hint: there is overlap between the two squences!)

   The only way to get HHH before TTT is to get it in the first three flips, so 1/8. If you flip a T at any point in time, then any string of heads must be preceded by a tail at some point in time, so HHH (if not the first three flips) must be preceded by THH.

4. Flip a coin until you either have 1 more tail than heads (lose) or have 3 more heads than tails (win). What is your chance of winning?

   If you flip a tail, you lose. If you flip a head, you get to a symmetrical position and so have 1/2 chance of winning from there. The chance of winning overall is therefore 1/4.

5. An urn has 837 red balls, 3 blue balls, and 2 green balls. You take balls out one at a time until you draw a blue (win) or a green (lose). What is your chance of winning?

   Ignore the red. . . the chance of victory is 3/5. You can imagine that the red balls are outside the urn and drawing red equates to "missing" the urn entirely.