

Chapter 7.2: Probability

Monday, July 27

Summary

- Bernoulli trial: 0/1 outcome with p chance of success, $0 \leq p \leq 1$.
- Chance of k successes in n trials: $\binom{n}{k} p^k (1-p)^{n-k}$.

Sampling With and Without Replacement

1. There are 99 red balls and 1 black ball in an urn. What is the chance of drawing only red balls if you draw 10 balls with replacement?
2. What is the chance of drawing only red balls if you draw 10 balls without replacement?
3. Compare your answers to the previous two questions and try to generalize the result.
4. (★) There are 5 red balls and 5 blue balls in an urn. You draw from the urn twice at random and would like to draw 1 red ball and 1 blue ball. Should you sample with or without replacement?
5. There are 2 red balls, 3 green balls, and 2 blue balls in an urn. You will keep drawing balls at random until you get either a green (win) or a blue (lose). Is it better to sample with or without replacement?

Bernoulli Trials

1. A coin has a $3/10$ chance of landing on heads. What is the probability that flipping the coin 10 times will result in 2 heads? 3 heads? 4 heads?
2. Which is more likely: that a coin ($p(\text{heads}) = 0.6$) will land on heads 5 times out of 10 or that a coin ($p(\text{heads}) = 0.4$) will land on heads 5 times out of 10?
3. (★) Which is more likely: that a fair coin will land on heads 5 times out of 10 or that a coin with a $3/5$ chance of landing on heads will land on heads 6 times out of 10?
4. Find a generalization of your answer to the above question and prove it.
5. Prove that $\binom{n}{k} \leq \binom{n}{k+1}$ if $k < n/2$ and $\binom{n}{k} \geq \binom{n}{k+1}$ otherwise. What does this say about the relative probabilities of getting 65 and 70 heads out of 100 with a fair coin?

Duels

1. (★) Two tennis players are currently tied in a game and will continue to play until one of them leads the other by 2 points. If player A has a 55% chance of winning any given point and if point outcomes are independent, what is the chance that A will win the game?
2. A and B wish to have a duel but only own one revolver between the two of them. They arrange the duel as follows: The revolver has 6 chambers and only a single bullet. A spins the revolver and fires at B , then B does the same to A . They continue until one of them successfully fires the revolver. What is the chance that A will win?
3. (Mosteller) A , B , and C are to fight a three-cornered pistol duel. All know that A has an 0.3 chance of hitting his target in any given shot, C has an 0.5 chance, and B never misses. They will go in order A , B , and C and fire at the target of their choice until there is only one person left (they may also deliberately miss). What should A 's strategy be?
4. (Mosteller) The rules for the game of craps are as follows: roll two dice. 7 or 11 = win, 2, 3, or 12 = lose. Any other number becomes your "point." If the first throw is a "point," keep rolling the dice until you win by rolling your point again or lose by rolling a 7. What is your chance of winning?

Challenge

1. Prove that the probability of getting k heads out of n flips with a coin that lands on heads with probability p increases with k up to (n/p) and then decreases.
2. (Mosteller) Urn A has 2 red balls and 1 black, and Urn B has 101 red and 100 black. An urn is chosen at random, and you win a prize if you correctly name the urn on the basis of the evidence of two balls drawn from it. After the first ball is drawn and its color reported, you may decide whether to replace the ball before the second drawing. What strategy maximizes your chance of victory?

Problems from Rosen

7.2: 19, 23, 29, 33, 35, 38