

Chapter 7.2: Probability

Monday, July 27

Summary

- Bernoulli trial: 0/1 outcome with p chance of success, $0 \leq p \leq 1$.
- Chance of k successes in n trials: $\binom{n}{k} p^k (1-p)^{n-k}$.

Sampling With and Without Replacement

1. There are 99 red balls and 1 black ball in an urn. What is the chance of drawing only red balls if you draw 7 balls with replacement?

$(99/100)^7$

2. What is the chance of drawing only red balls if you draw 10 balls without replacement?

$93/100$.

3. Compare your answers to the previous two questions and try to generalize the result.

$(99/100)^7 > 93/100$. The generalization is $(1 - 1/n)^k > 1 - k/n$.

4. (★) There are 5 red balls and 5 blue balls in an urn. You draw from the urn twice at random and would like to draw 1 red ball and 1 blue ball. Should you sample with or without replacement?

If you sample with replacement the chance is $1/2$. If you sample without replacement the chance is $25/\binom{10}{2} = 25/45 = 5/9$. You should sample without replacement.

5. There are 2 red balls, 3 green balls, and 2 blue balls in an urn. You will keep drawing balls at random until you get either a green (win) or a blue (lose). Is it better to sample with or without replacement?

It won't matter. Either way, you have a $3/5$ chance of drawing a green ball first.

If you sample with replacement then the probability of drawing green before blue is $P = 3/7 + (2/7)P$, giving the answer $P = 3/5$.

If you sample without replacement, the probability of drawing green before blue is $p(G) + p(RG) + p(RRG) = \frac{3}{7} + \frac{2}{7} \frac{3}{6} + \frac{2}{7} \frac{1}{6} \frac{3}{5} = \frac{4}{7} + \frac{1}{35} = \frac{3}{5}$.

Bernoulli Trials

1. A coin has a $3/10$ chance of landing on heads. What is the probability that flipping the coin 10 times will result in 2 heads? 3 heads? 4 heads?

Respective chances are .23, .27, and .20.

2. Which is more likely: that a coin ($p(\text{heads}) = 0.6$) will land on heads 5 times out of 10 or that a coin ($p(\text{heads}) = 0.4$) will land on heads 5 times out of 10?

The probabilities are the same, by symmetry.

3. (★) Which is more likely: that a fair coin will land on heads 5 times out of 10 or that a coin with a $3/5$ chance of landing on heads will land on heads 6 times out of 10?

Probabilities .246, .251, respectively.

4. Find a generalization of your answer to the above question and prove it.

One generalization could be that the function $\binom{n}{k}(k/n)^k(1-k/n)^{n-k}$ hits a minimum at $k/n = 0.5$ and is increasing thereafter.

One potential proof: Say $k/n \geq 0.5$, then want to prove

$$\begin{aligned}\binom{n}{k}(k/n)^k(1-k/n)^{n-k} &\leq \binom{n}{k+1}((k+1)/n)^{(k+1)}(1-(k+1)/n)^{n-(k+1)} \\ \frac{n!}{k!(n-k)!}k^k(n-k)^{n-k} &\leq \frac{n!}{(k+1)!(n-k-1)!}(k+1)^{k+1}(n-k-1)^{n-k-1} \\ k^k(n-k)^{n-k-1} &\leq (k+1)^k(n-k-1)^{n-k-1} \\ \left(1 + \frac{1}{n-k-1}\right)^{n-k-1} &\leq \left(1 + \frac{1}{k}\right)^k\end{aligned}$$

This last statement is true because $k \geq n-k-1$ (since $k \geq (n-1)/2$) and $f(n) = (1+1/n)^n$ is an increasing function (why?)

5. Prove that $\binom{n}{k} \leq \binom{n}{k+1}$ if $k < n/2$ and $\binom{n}{k} \geq \binom{n}{k+1}$ otherwise. What does this say about the relative probabilities of getting 65 and 70 heads out of 100 with a fair coin?

Just handle the first case by working backwards—the second will follow by symmetry:

$$\begin{aligned}\binom{n}{k} &\leq \binom{n}{k+1} \\ \frac{n!}{k!(n-k)!} &\leq \frac{n!}{(k+1)!(n-k-1)!} \\ k+1 &\leq n-k \\ k &\leq (n-1)/2\end{aligned}$$

Duels

1. (★) Two tennis players are currently tied in a game and will continue to play until one of them leads the other by 2 points. If player A has a 55% chance of winning any given point and if point outcomes are independent, what is the chance that A will win the game?

If the match does not end after 2 rounds, then A and B have both won a single match and are back where they started. So assume the match ends after 2 rounds: the odds that A has won both matches are $.55^2 : .45^2$, so the probability is $\frac{.55^2}{.55^2 + .45^2} = .599$.

2. A and B wish to have a duel but only own one revolver between the two of them. They arrange the duel as follows: The revolver has 6 chambers and only a single bullet. A spins the revolver and fires at B, then B does the same to A. They continue until one of them successfully fires the revolver. What is the chance that A will win?

If the duel does not end after 2 rounds then A and B are back where they started: both are alive and A has the revolver. The chance that A wins in one shot are $1/6$ and the chance that B wins in his first shot are $(5/6)(1/6)$, so the odds that A wins are $1 : (5/6)$, and the probability that A wins are $\frac{1}{1+5/6} = 6/11$.

3. (Mosteller) A, B, and C are to fight a three-cornered pistol duel. All know that A has an 0.3 chance of hitting his target in any given shot, C has an 0.5 chance, and B never misses. They will go in order A, B, and C and fire at the target of their choice until there is only one person left (they may also deliberately miss). What should A's strategy be?

A should fire his shot into the air. Then B will kill C and A will have the first shot against B, with an 0.3 chance of winning.

If shoots at B and kills him, then C gets the first shot against A. C has an 0.5 chance of victory in 1 round and A has an $(0.5)(0.3) = 0.15$ chance of victory in his first shot, so the chance that A wins in this case are $\frac{0.15}{0.15+0.5} = .23$.

4. (Mosteller) The rules for the game of craps are as follows: roll two dice. 7 or 11 = win, 2, 3, or 12 = lose. Any other number becomes your "point." If the first throw is a "point," keep rolling the dice until you win by rolling your point again or lose by rolling a 7. What is your chance of winning?

Chance of victory in the first round = $8/36$, chance of defeat = $4/36$. After this, we have to break down the possibilities of victory by what the point is:

- (a) 4 or 10: $6/36$ chance of rolling either, then $1/3$ chance of rolling again before the 7.
- (b) 5 or 9: $8/36$ chance of rolling, then $2/5$ chance of rolling again before the 7.
- (c) 6 or 8: $10/36$ chance of rolling, then $5/11$ chance of rolling again before the 7.

Total chance of victory: $8/36 + (1/6)(1/3) + (8/36)(2/5) + (10/36)(5/11) = .4929 \dots$ The probability of victory is just shy of $1/2$.

Challenge

1. Prove that the probability of getting k heads out of n flips with a coin that lands on heads with probability p increases with k up to (n/p) and then decreases.

Work backwards from the result... the following statements are all equivalent:

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &\leq \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1} \\ \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} &\leq \frac{n!}{(k+1)!(n-k-1)!} p^{k+1} (1-p)^{n-k-1} \\ (k+1)(1-p) &\leq (n-k-1)p \\ 1+k-p-kp &\leq np-kp-p \\ k &\leq np-1 \end{aligned}$$

2. (Mosteller) Urn A has 2 red balls and 1 black, and Urn B has 101 red and 100 black. An urn is chosen at random, and you win a prize if you correctly name the urn on the basis of the evidence of two balls drawn from it. After the first ball is drawn and its color reported, you may decide whether to replace the ball before the second drawing. What strategy maximizes your chance of victory?

Problems from Rosen

7.2: 19, 23, 29, 33, 35, 38