

Chapter 6.3: Permutations and Combinations

Tuesday, July 21

Summary

- r-permutations of n with repetition: n^r
- r-combinations of n with repetition: stars and bars. . . $\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$
- permutations with repetition: $\frac{n!}{k_1!k_2!\cdots k_m!}$ if there are k_i identical elements of type i .
- Distinguishable objects in distinguishable boxes so that there are k_i objects in the i -th box: same as “permutations with repetition.”
- Indistinguishable objects in distinguishable boxes: stars and bars again.
- Indistinguishable objects in indistinguishable boxes: partitions. . .

Stars and Bars

1. How many ways can you buy 8 fruit if your options are apples, bananas, pears, and oranges?
2. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?
3. How many ways can you give 10 cookies to 4 friends if each friend gets either 2 or 3 cookies? (No stars and bars required)
4. How many tuples of integers (x_1, x_2, x_3, x_4) are there such that $0 \leq x_1 < x_2 < x_3 < x_4 < 10$? (You do not need stars and bars for this—pick the integers, then sort them.)
5. How many tuples of integers (x_1, x_2, x_3, x_4) are there such that $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 10$?
6. How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, x_3 are non-negative integers? (You have 10 objects to distribute among 3 variables.)
7. How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, x_3 are integers greater than or equal to 2?

Multinomials

1. How many ways can you rearrange the letters in ABRACADABRA?
2. You have a 52-card deck. How many ways can you deal 5 cards to each of 6 players?
3. How many ways can you deal 13 cards to each of 4 players?
4. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (ace-2-3-...-king)? (No multinomials required.)

Partitions

1. How many ways to put 7 balls in 4 identical bins?
2. How many ways to put 7 balls in 7 identical bins?

Challenge

1. You start at the bottom left corner of a triangle with n circles to a side. You make $n - 1$ moves, and have 3 options for each move: go directly to the right, go up and to the right, or stay put. If the order in which you make the moves does not matter, show that every circle on the triangle corresponds to exactly one sequence of moves. Use the stars and bars method to show that the n -th triangular number is $\binom{n+1}{2}$.
2. Adapt your solution to the above problem to show that the number of spheres in a pyramid with n spheres on each side is $\binom{n+2}{3}$.
3. There are n white balls on a straight line. How many ways are there to paint some (or none) of the balls green so that no two adjacent balls are green?

Problems from Rosen

6.5: 1-10 for warmup, 16, 20, 21, 35, 39, 40, 45, 46