Chapter 6.3: Permutations and Combinations
Tuesday, July 21

Summary

- r-permutations of n with repetition: n^r

- r-combinations of n with repetition: stars and bars... $\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$

- permutations with repetition: $\dfrac{n!}{k_1! k_2! \cdots k_m!}$ if there are k_i identical elements of type i.

- Distinguishable objects in distinguishable boxes so that there are k_i objects in the i-th box: same as "permutations with repetition."

- Indistinguishable objects in distinguishable boxes: stars and bars again.

- Indistinguishable objects in indistinguishable boxes: partitions...

Stars and Bars

1. How many ways can you buy 8 fruit if your options are apples, bananas, pears, and oranges?

   8 choices from 4 options with repetition, so the number of ways is $\binom{8+4-1}{4-1} = \binom{11}{3} = 165$.

2. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

   Give out 4 "necessary" cookies, then count the number of ways to give 6 cookies to 4 friends if some can get no cookies. 6 choices from 4 options with repetition, so the number of ways is $\binom{6+4-1}{4-1} = \binom{9}{3} = 84$.

3. How many ways can you give 10 cookies to 4 friends if each friend gets either 2 or 3 cookies? (No stars and bars required)

   2 friends must get 2 cookies and 2 must get 3 cookies, so just choose which 2 friends get 3 cookies: the number of ways is $\binom{4}{2} = 6$.

4. How many tuplets of integers (x_1, x_2, x_3, x_4) are there such that $0 \le x_1 < x_2 < x_3 < x_4 < 10$? (You do not need stars and bars for this–pick the integers, then sort them.)

   This is the same as the number of sets of 4 distinct numbers $\{x_1, x_2, x_3, x_4\}$ from between 0 and 9. The number of tuplets is $\binom{10}{4}$.

5. How many tuplets of integers (x_1, x_2, x_3, x_4) are there such that $0 \le x_1 \le x_2 \le x_3 \le x_4 \le 10$?

   Imagine each x_i as a "choice" of an integer. The problem is then to choose 4 integers from 11 possibilities with repetition allowed. The number of ways is $\binom{11+4-1}{11-1} = \binom{14}{10} = \binom{14}{4} = 1001$.

6. How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, x_3 are non-negative integers? (You have 10 objects to distribute among 3 variables.)

You are choosing 10 "ones" to give to 3 variables with repetition, so the number of ways is $\binom{10+3-1}{3-1} = \binom{12}{2} = 66$.

Note also: since the choice for x_3 is automatic once x_1 and x_2 are picked, this is the same as the number of non-negative solutions in two variables to the inequality $x_1 + x_2 \leq 10$.

7. How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, x_3 are integers greater than or equal to 2?

   Letting $x_i = y_i + 2$ for $i = 1, 2, 3$ gives the new problem of finding the number of non-negative solutions to the equation $y_1 + y_2 + y_3 = 4$. This is a problem of 4 choices from 3 variables with repetition, so the number of solutions is $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$.

Multinomials

1. How many ways can you rearrange the letters in ABRACADABRA?

   $\dfrac{11!}{5!2!2!1!1!} = 83160.$

2. You have a 52-card deck. How many ways can you deal 5 cards to each of 6 players?

   $\dfrac{52!}{(5!)^6 \cdot 22!}$

3. How many ways can you deal 13 cards to each of 4 players?

   $\dfrac{52!}{13!13!13!13!}$

4. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (ace-2-3-...-king)? (No multinomials required.)

   Approach this problem value by value: there are 24 ways to distribute the aces to the 4 players (4!), 24 ways to distribute the twos, and so on, so the number of ways to deal the cards in this manner is $(4!)^{13} = 24^{13}$.

Partitions

1. How many ways to put 7 balls in 4 identical bins?

   There are 11 ways in all:

   $$7 + 0 + 0 + 0 = 7$$
   $$6 + 1 + 0 + 0 = 7$$
   $$5 + 2 + 0 + 0 = 7$$
   $$5 + 1 + 1 + 0 = 7$$
   $$4 + 3 + 0 + 0 = 7$$
   $$4 + 2 + 1 + 0 = 7$$
   $$4 + 1 + 1 + 1 = 7$$
   $$3 + 3 + 1 + 0 = 7$$
   $$3 + 2 + 2 + 0 = 7$$
   $$3 + 2 + 1 + 1 = 7$$
   $$2 + 2 + 2 + 1 = 7$$

2. How many ways to put 7 balls in 7 identical bins?

   15 ways total: all of the above ways, plus the following ones that would require more than 4 bins:

   $$3 + 1 + 1 + 1 + 1 = 7$$
   $$2 + 2 + 1 + 1 + 1 = 7$$
   $$2 + 1 + 1 + 1 + 1 + 1 = 7$$
   $$1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$$

Challenge

1. You start at the bottom left corner of a triangle with n circles to a side. You make $n-1$ moves, and have 3 options for each move: go directly to the right, go up and to the right, or stay put. If the order in which you make the moves does not matter, show that every circle on the triangle corresponds to exactly one sequence of moves. Use the stars and bars method to show that the n-th triangular number is $\binom{n+1}{2}$.

2. Adapt your solution to the above problem to show that the number of spheres in a pyramid with n spheres on each side is $\binom{n+2}{3}$.

3. There are n white balls on a straight line. How many ways are there to paint some (or none) of the balls green so that no two adjacent balls are green?

Problems from Rosen

6.5: 1-10 for warmup, 16, 20, 21, 35, 39, 40, 45, 46