Chapter 6.3: Permutations and Combinations

Tuesday, July 21

Summary

- Pascal's Identity: $\dbinom{n+1}{k} = \dbinom{n}{k} + \dbinom{n}{k-1}$

- Binomial Theorem: $(x+y)^n = \displaystyle\sum_{j=0}^{n} \dbinom{n}{j} x^{n-j} y^j$

- Vandermonde's Identity: $\dbinom{m+n}{r} = \displaystyle\sum_{k=0}^{r} \dbinom{m}{r-k}\dbinom{n}{k}$

- Multinomial Theorem: $(x_1 + x_2 + \cdots + x_m)^n = \displaystyle\sum_{n_1+n_2+\cdots n_m=n} \frac{n!}{n_1! n_2! \cdots n_m!} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$

Binomial Theorem

1. You flip 5 coins. How many ways are there to get an even number of heads?

2. Evaluate using the Binomial Theorem: $\displaystyle\sum_{i=0}^{10} \dbinom{10}{i} 4^{10-i}$.

3. (★) How many ways to rearrange the letters in COUSCOUS?

4. Prove algebraically: if $1 \leq k \leq n$ then $\dbinom{n}{k} \geq \left(\dfrac{n}{k}\right)^k$.

5. You have 3 red hats and 4 green hats and 5 blue hats and 12 friends. How many ways to give each friend a hat?

Combinatorial Proofs

1. (★) Show that if n is a positive integer then $\dbinom{2n}{2} = 2\dbinom{n}{2} + n^2$, by combinatorial proof and by algebraic manipulation. (Hint: there are n boys and n girls. If you want to pick 2 people for a team, break down by the number of girls you pick.)

Pascal's Triangle

1. Show that $\dfrac{4^n}{2n+1} \leq \dbinom{2n}{n} \leq 4^n$.

2. ($\bigstar$) Verify for $n = 0, 1, 2, 3, 4$ the relation $\displaystyle\sum_{j+k=n} \dbinom{j}{k} = f_n$, the n-th Fibonacci number. Draw a picture illustrating this identity on Pascal's Triangle, then prove by induction.

Challenge

1. If $1 \leq k \leq n$ then $\dbinom{n}{k} < \left(\dfrac{en}{k}\right)^k$ (use Binomial Theorem and the fact that $e^x > 1 + x$ for $x \neq 0$).

2. If $1 \leq k \leq n$ then $\displaystyle\sum_{j=0}^{k} \dbinom{n}{j} < \left(\dfrac{en}{k}\right)^k$.

Problems from Rosen

6.4: 14, 19, 20, 21, 22, 24, 29, 32, 33. The book has lots of good exercises with making combinatorial arguments.