Chapter 6.3: Permutations and Combinations
Tuesday, July 21

Summary

• Pascal’s Identity: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \)

• Binomial Theorem: \((x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\)

• Vandermonde’s Identity: \( \binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k} \)

• Multinomial Theorem: \( (x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1 + n_2 + \cdots + n_m = n} \frac{n!}{n_1! n_2! \cdots n_m!} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m} \)

Binomial Theorem

1. You flip 5 coins. How many ways are there to get an even number of heads?

2. Evaluate using the Binomial Theorem: \( \sum_{i=0}^{10} \binom{10}{i} 4^{10-i} \).

3. (⋆) How many ways to rearrange the letters in COUSCOUS?

4. Prove algebraically: if \( 1 \leq k \leq n \) then \( \binom{n}{k} \geq \frac{n!}{k!} \).

5. You have 3 red hats and 4 green hats and 5 blue hats and 12 friends. How many ways to give each friend a hat?

Combinatorial Proofs

1. (⋆) Show that if \( n \) is a positive integer then \( \binom{2n}{2} = 2 \binom{n}{2} + n^2 \), by combinatorial proof and by algebraic manipulation. (Hint: there are \( n \) boys and \( n \) girls. If you want to pick 2 people for a team, break down by the number of girls you pick.)
Pascal’s Triangle

1. Show that \( \frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n \).

2. (★) Verify for \( n = 0, 1, 2, 3, 4 \) the relation \( \sum_{j+k=n} \binom{j}{k} = f_n \), the n-th Fibonacci number. Draw a picture illustrating this identity on Pascal’s Triangle, then prove by induction.

Challenge

1. If \( 1 \leq k \leq n \) then \( \binom{n}{k} < \left( \frac{e^n}{k} \right)^k \) (use Binomial Theorem and the fact that \( e^x > 1 + x \) for \( x \neq 0 \)).

2. If \( 1 \leq k \leq n \) then \( \sum_{j=0}^{k} \binom{n}{j} < \left( \frac{e^n}{k} \right)^k \).

Problems from Rosen

6.4: 14, 19, 20, 21, 22, 24, 29, 32, 33. The book has lots of good exercises with making combinatorial arguments.