Chapter 6.3: Permutations and Combinations

Tuesday, July 21

Summary

- Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- Binomial Theorem: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$
- Vandermonde's Identity: $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$
- Multinomial Theorem: $(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots n_m = n} \frac{n!}{n_1! n_2! \cdots n_m!} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$

Binomial Theorem

1. You flip 5 coins. How many ways are there to get an even number of heads?

   $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16$. Also, by an earlier identity the number of ways to get an even number of heads is the same as the number of ways to get an odd number, so divide the total options by 2 to get $32/2 = 16$.

2. Evaluate using the Binomial Theorem: $\sum_{i=0}^{10} \binom{10}{i} 4^{10-i}$.

   $$\sum_{i=0}^{10} \binom{10}{i} 4^{10-i} = (1+4)^{10} = 5^{10}$$

3. (★) How many ways to rearrange the letters in COUSCOUS?

   There are 8! ways to rearrange the letters, then divide by 2^4 for the 4 pairs of letters (each letter can be swapped with its partner to get an identical arrangement), so $\frac{8!}{2^4} = 2520$ ways in all.

4. Prove algebraically: if $1 \leq k \leq n$ then $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$.

   Work backwards to figure out this one:

   $$\begin{aligned} \binom{n}{k} \geq \left(\frac{n}{k}\right)^k &\Leftrightarrow \frac{n!}{k!(n-k)!} \geq \frac{n^k}{k^k} \\ &\Leftrightarrow \frac{k^k}{k!} \geq \frac{n^k}{n(n-1)\cdot(n-k+1)} \\ &\Leftrightarrow \prod_{i=0}^{k-1} \frac{k}{k-i} \geq \prod_{i=0}^{k-1} \frac{k}{k-i} \end{aligned}$$

   The last statement is true if $\frac{k}{k-i} \geq \frac{n}{n-i}$ for $1 \leq i \leq k-1$, which is is because $n \geq k$. Therefore the original statement is true.

5. You have 3 red hats and 4 green hats and 5 blue hats and 12 friends. How many ways to give each friend a hat?

   The number of ways is $\frac{12!}{3!4!5!} = 27720$.

Combinatorial Proofs

1. (★) Show that if n is a positive integer then $\binom{2n}{2} = 2\binom{n}{2} + n^2$, by combinatorial proof and by algebraic manipulation. (Hint: there are n boys and n girls. If you want to pick 2 people for a team, break down by the number of girls you pick.)

   Combinatorial: If there are n boys and n girls and you want to pick 2 of them (so, $\binom{2n}{2}$ options), there are $\binom{n}{2}$ ways to pick 2 boys and $\binom{n}{2}$ ways to pick 2 girls and n^2 ways to pick 1 boy and 1 girl.

   Algebraic: $\binom{2n}{2} = \frac{2n(2n-1)}{2} = 2n^2 - n = n^2 + (n^2 - n) = n^2 + 2\binom{n}{2}$.

Pascal's Triangle

1. Show that $\dfrac{4^n}{2n+1} \leq \dbinom{2n}{n} \leq 4^n$.

   We know that the sum of all elements in row 2n of Pascal's Triangle is $2^{2n} = 4^n$, so $\dbinom{2n}{n}$ must be less than this. But there are (2n+1) elements in the row and $\dbinom{2n}{n}$ is the largest of these elements (prove this statement!) so it must be bigger than the average, which is $\frac{4^n}{2n+1}$. This puts a pretty reasonable bound on the size of $\dbinom{2n}{n}$.

2. (★) Verify for $n = 0, 1, 2, 3, 4$ the relation $\displaystyle\sum_{j+k=n} \binom{j}{k} = f_n$, the n-th Fibonacci number. Draw a picture illustrating this identity on Pascal's Triangle, then prove by induction.

   The picture would involve diagonals moving leftward across the triangle (which are more flat than the sides of the triangle itself). Visually, the relation should hold because the sum of the elements in two diagonals (using Pascal's Identity) should lead to the next diagonal.

   Proof by induction: For the base case, we have $\dbinom{0}{0} = 1 = f_0$ and $\dbinom{1}{0} = 1 = f_1$.

   Inductive step: Suppose the formula holds for n and $n+1$; we want to show that it then also holds for $n+2$.

$$
\begin{aligned}
\sum_{j+k=n+2} \binom{j}{k} &= \sum_{j+k=n+2} \binom{j-1}{k} + \binom{j-1}{k-1} \\
&= \sum_{j+k=n+2} \binom{j-1}{k} + \sum_{j+k=n+2} \binom{j-1}{k-1} \\
&= \sum_{r+k=n+1} \binom{r}{k} + \sum_{r+l=n} \binom{r}{l} \\
&= f_{n+1} + f_n \\
&= f_{n+2}
\end{aligned}
$$

   The change in the third line comes from substituting $r = j-1, l = k-1$ in the relevant sums.

Challenge

1. If $1 \leq k \leq n$ then $\dbinom{n}{k} < \left(\dfrac{en}{k}\right)^k$ (use Binomial Theorem and the fact that $e^x > 1 + x$ for $x \neq 0$).

2. If $1 \leq k \leq n$ then $\displaystyle\sum_{j=0}^{k} \binom{n}{j} < \left(\frac{en}{k}\right)^k$.

Problems from Rosen

6.4: 14, 19, 20, 21, 22, 24, 29, 32, 33. The book has lots of good exercises with making combinatorial arguments.