

Chapter 6.3: Permutations and Combinations

Tuesday, July 21

Summary

- $P(n, k) = \text{"k-permutations of n"} = n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}$
- $\binom{n}{k} = \text{"n choose k"} = \frac{n(n-1) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$
- $\binom{n}{k} = \binom{n}{n-k}$
- Combinatorial Proofs 1: If A and B are finite sets and $f : A \rightarrow B$ is a bijection, then $|A| = |B|$.
- Combinatorial Proofs 2: Any two (correct) ways of counting the elements in a set will yield the same answer.

Warmup

1. (★) Six people are in a club. How many ways to choose three executive members? How many ways are there to choose a president and two vice presidents? Why is the second number divisible by the first?

The number of ways to choose three executive members is $\binom{6}{3} = 20$. The number of ways to choose a president and two vice presidents is $6 \cdot \binom{5}{2} = 60$.

The second is divisible by the first because we have another process for choosing a president and two vice presidents that involves using the first process: first choose three people to be executives. Then choose one of those three to be president, and the other two will be VP's. This gives a total of $\binom{6}{3} \cdot \binom{3}{1} = 60$ options.

2. Ten people are in a room, and everyone shakes everyone else's hand exactly once. How many handshakes were there?

There were $\binom{10}{2} = 45$ handshakes.

Inclusion-Exclusion

1. How many 5-digit numbers start with a 1 or end with a 0 (or both)?

$10^4 = 10000$ numbers start with a 1 and $9 \cdot 10^3 = 9000$ end with a 0 (because numbers cannot start with a 0). $10^3 = 1000$ numbers start with a 1 and end with a zero, so the total number is $10000 + 9000 - 1000 = 18000$.

Alternate: This is the disjoint union of the set of numbers that start with a 1 ($1 \cdot 10^4$). and those that end with a 0 but DO NOT start with a 1 ($8 \cdot 10^3 \cdot 1$). So, 18000 in all.

2. (★) How many 5-digit numbers contain at least one 1 and at least one 3? (Hint: what is the complement of this set in the set of 5-digit numbers?)

There are $9 \cdot 10^4$ 5-digit numbers in all. $8 \cdot 9^4$ of them do not contain a 1 and the same number do not contain a 3. $7 \cdot 8^4$ contain neither. Thus the number of 5-digit numbers that do not have a 1 OR do not have a 3 is $8 \cdot 9^4 + 8 \cdot 9^4 - 7 \cdot 8^4 = 76304$. This means that the number of numbers with at least one 1 and at least one 3 is $90000 - 76304 = 13696$.

Tricks With Counting

1. A coin is flipped seventy-three times. How many possible outcomes have more heads than tails? (Hint: what happens if you swap the words “heads” and “tails”?)

Exactly half of the outcomes have more heads than tails, so the total number with more heads than tails is $2^{73}/2$.

2. How many ways are there to place four red balls and two blue balls around a circular table?

Three. If we tried using the division formula and calculating $\binom{6}{2}/6 = 15/6 = 2.5$, we wouldn't even get an integer! The reason is because not every arrangement has 6 distinct rotations: if we put the two blue balls on opposite sides of the table, this arrangement has only 3 distinct rotations. (This has to do with the fact that there is a repeating pattern red-red-blue of length 3, which divides 6 evenly. Lesson: be careful.)

3. (★) How many distinct permutations of the letters CHECKMATE contain the string TEAM? (Hint: treat the string “TEAM” as a solid block and go from there.)

This is the same as the number of permutations of (TEAM)CCHEK, where (TEAM) is treated as a solid block. The number of way is then $\binom{6}{2} \cdot 4! = 360$.

Combinatorial Proofs

1. People go around in a room shaking hands with one another (though it is not necessarily true that everybody shakes everybody else's hand). Prove that the number of people who shake an *odd* number of hands is *even*. (Count the number of person-handshakes, first by person and then by handshake.)

Since each handshake counts for two person-handshakes, the number of person-handshakes is even.

Then count the person handshakes as the number of handshakes over the people who shook an even number of hands plus the number of handshakes over an people who shook an odd number of hands. The first number is even and the total is even, so the number of handshakes from people who shook an odd number of hands must be even as well. Since the number of hands shaken per person is odd, the number of people must be even.

2. (★) Find an argument by bijection that the number of subsets of $\{1 \dots n\}$ of even size is equal to the number of subsets of odd size. (Is 1 in your set, or isn't it?)

Describe a function f as follows: If $a \in S$, then $f(S) = S - \{a\}$. If $a \notin S$, then $f(S) = S \cup \{a\}$. Note that $f(S) \neq S$ and that $f(f(S)) = S$, so this function f has the effect of pairing every set with a different partner. Furthermore, if $|S|$ is odd then $|f(S)|$ is even and vice versa. So every set of odd size is paired with one of even size, and so the number of such sets must be the same.

3. Find an argument by bijection that $\binom{2n}{n}$ is even whenever $n \geq 1$. (Let S be the set of all n -combinations of $\{1 \dots 2n\}$ and find a bijection from S to itself. If no sets are left fixed under this function, why does that prove that the size is even?)

Let $U = \{1 \dots 2n\}$ and let C be the set of n -combinations of U . For $S \in C$, define the function f by $f(S) = \overline{S}$. It is easy to check that $f(S) \neq S$ and that $f(f(S)) = S$ for all S . Therefore f pairs each of the elements of C with a unique partner, and so the number of elements in C (thus $\binom{2n}{n}$) is even.

The statement $n \geq 1$ was required so that our set of n -combinations would not contain only the empty set.

Suggested From Rosen

6.3: 19, 21, 23, 24, 33, 34, 35