Chapters 6.1-6.2: Counting
Monday, July 20

- How many three-letter initials can people have?

  There are three letters with 26 choices for each letter, so 26^3 options in total.

- If a password consists of a two-digit number followed by a five-character string of English letters, how many possible passwords are there?

  Assuming it is okay for the number to start with a zero, then there are $10^2 \cdot 26^5$ options. If not, then there are $9 \cdot 10 \cdot 26^5$ options.

- A palindrome is a string that reads the same forwards and backwards (e.g. "1771") How many 4-digit numbers are palindromes? How many 5-digit numbers?

  Once we set the first two digits, the last two are automatically determined since the number must be a palindrome. The number cannot start with a zero, so there are $9 \cdot 10 = 90$ four-digit palindromes and $9 \cdot 10 \cdot 10$ five-digit palindromes.

- A multiple choice test has 6 questions with 4 possible answers each. Students taking the test also have the option to leave answers blank. How many students must take the test in order to guarantee that at least two students have the exact same answers?

  There are 6 questions and 5 ways to answer (or not answer) each question, making for $5^6 = 15625$ ways to respond to the test. Therefore, 15626 students need to take the test in order to guarantee that two students will have the exact same answers.

- How many numbers between 1 and 1000

  1. Are divisible by both 7 and 11?
     $\lfloor 1000/7 \rfloor = 142$ numbers are divisible by 7, $\lfloor 1000/11 \rfloor = 90$ are divisible by 11, and $\lfloor 1000/77 \rfloor = 12$ are divisible by 77 (these are the only ones divisible by 7 and 11).

  2. Are divisible by either 7 or 11?
     Using inclusion-exclusion, the number divisible by 7 or 11 is 142 + 90 - 12 = 220.

  3. Are divisible by 7 but not by 11?
     The number divisible by 7 but not 11 is 142 - 12 = 130 (draw a Venn diagram to justify this).

  4. Are divisible by neither 7 nor 11?
     This is the same as NOT (divisible by 7 or 11), so the number of such elements is 1000 - 220 = 780.

- 36 students go to a hot dog stand and order hot dogs. Every student orders at least one topping. You have the following information about their topping choices:

  1. 18 ask for mustard.
  2. 21 ask for onions.
  3. 18 ask for relish.
  4. 8 ask for mustard but not onions.
  5. 31 ask for onions or relish (or both).
  6. 17 ask for exactly two toppins.
  7. 2 ask for all three toppings.

  How many students order exactly one topping? (Try making a Venn diagram. See also Chapter 8.5 of Rosen)

  Denote the numbers of students who get mustard ONLY, onion ONLY, and relish ONLY by M, O, and R, respectively. Those who get two ingredients are denoted by MO, MR, and OR, and those who get three by MOR. Then we are given the equations

1. $M + MO + MR + MOR = 18$
2. $O + MO + OR + MOR = 21$
3. $R + MR + OR + MOR = 18$
4. $M + MR = 8$
5. $O + R + MO + MR + OR + MOR = 31$
6. $MO + MR + OR = 17$
7. $MOR = 2$
8. $M + O + R + MO + MR + OR + MOR = 36$

Combining (5) and (8) gives $M = 5$, and combining (5), (6), and (7) gives $O + R = 12$. Therefore $M + O + R = 17$ students get exactly one ingredient.

- Use a tree diagram to find the number of subsets of $\{3, 7, 9, 11, 24\}$ such that the sum of the elements in the subset is less than 28.

  There are at least two ways to make the decision tree: first, one can split by which number is the smallest element of the set (leading to 6 branches: none/3/6/9/11/24). Second, one could pick an element and ask whether it is in the set (2 branches: yes/no).

  For example: if 24 is in the set, then the sum of the other elements must be less than four. This leads to only two options, $\{3, 24\}$ and $\{24\}$.

  Otherwise, 24 is not in the set and we have the new problem of finding the number of subsets of $\{3, 7, 9, 11\}$ whose sum is less than 28. There is a shortcut here: the sum of all four elements is 30 (too large), but the sum of any three elements is at most 27. Therefore ANY subset except for the whole one will work, and there are $2^4 - 1 = 15$ such options.

  The total number of options is therefore 17.

- If we make a 4-sided die out of a tetrahetron (4 faces, all equilateral triangles), then how many possible arrangements of the numbers are there? How many possible arrangements are there on a 6-sided die?

  There are 12 symmetries of a tetrahedron: we can pick one of the 4 vertices to be the top vertex, then spin the tetrahedron 3 ways. There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to put the 4 numbers on the faces, and so only $24/12 = 2$ of those ways are distinct.

  On the 6-sided die there are 24 symmetries (pick a face to be the top face, then spin it around 4 ways). The total number of distinct dice is therefore $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1/24 = 30$.

  Bonus question: how many six-sided dice are there if the sum of opposite sides must be 7?

- Put an upper bound on the number of four-letter words in the English language. Try to make this bound as tight as possible by introducing rules about what letters words must have and what letter combinations are not possible. Try any of the following:

  1. Every word must have at least one vowel.

     This is equal to the total number of combinations minus the ones with no vowels, so $26^4 - 20^4$ (assuming "y" counts as a vowel, since it allows for words like "lynx"). Total: 296976

  2. "q" must always be followed by "u," then some vowel (with rare exceptions).

     Break into a decision tree: "q" can be the first or the second letter, or it might not appear at all.

     First option: $1 \cdot 1 \cdot 6 \cdot 26$

     Second option: $26 \cdot 1 \cdot 1 \cdot 6$

     Third option: $25 \cdot 25 \cdot 26 \cdot 26$ (the first two letters have only 25 options because we assumed that neither of them were "q"). Total: 422812

  3. If a word begins with "m," "n," "l,", "r,", or "h," the next letter must be a vowel.

     Break into a decision tree: either the first letter is one of those five or it isn't. The first option leads to $5 \cdot 6 \cdot 26 \cdot 26$ options and the second leads to $21 \cdot 26 \cdot 26 \cdot 26$ options. Add these two to get the total number of options. Total: 389376

     If we wanted to apply two or more of these rules at the same time then we could get a tighter bound, but the answers would be messier. The first bound is the lowest, but still very far from the true number (about 5500 according to one Scrabble dictionary, this will vary from source to source).

- (Harder) Let S be any positive integer. Use the Pigeonhole Principle to show that if we write the numbers S, $2S$, $3S$, ... $10S$ one underneath the other, then in every full column there will be at least one 0 or 9 appearing.

  Assume that the number 9 does not appear in a given column. Since there are 10 numbers in that column and (now) only 9 options of digits, at least two rows (say, $N \cdot S$ and $M \cdot S$ with $N < M$) have

the same digit. But then $(M - N)S$ will have a zero in that spot (or a 9 because of carry-over from the subtraction, but we already assumed that it didn't have a 9).

Example:

$$S = 1783$$
$$2S = 3556$$
$$3S = 5349$$
$$4S = 7132$$
$$5S = 8915$$
$$6S = 10698$$
$$7S = 12481$$
$$8S = 14264$$
$$9S = 16047$$
$$10S = 17830$$

For the tens place, both S and $7S$ have an 8, and as a consequence $6S$ has a 9 in the tens place. Both 4S and 10S have a 7 in the thousands place, and as a result 6S has a zero in that place.