

Chapter 5.2: Strong Induction

Tuesday, July 14

The Well-Ordering Property

1. Prove that 1 is the smallest natural number (The smallest natural number exists by the W-O.P. Let a be that number, and suppose $a < 1$. Then show that there is a smaller natural number, getting a contradiction).
2. Extend this property to \mathbb{Z} : show that if a set $S \subset \mathbb{Z}$ has a lower bound (some N such that $N \leq s$ for all $s \in S$) then S has a least element.
3. Find a subset of \mathbb{Z} with no lower bound.
4. Show that if a set $S \subset \mathbb{Z}$ or $S \subset \mathbb{N}$ has an upper bound, then S has a greatest element.

Strong Induction

1. Suppose $P(n)$ is a propositional function, that $P(0)$ and $P(1)$ are true, and that for any n if $P(n)$ and $P(n+1)$ are true then $P(n+2)$ is true. For what numbers n must $P(n)$ be true?
2. Suppose $P(n)$ is a propositional function, that $P(0)$ is true, and that for any n if $P(n)$ is true then $P(n+2)$ and $P(n+3)$ are true. For what numbers n must $P(n)$ be true?
3. Suppose $P(n)$ is a propositional function, that $P(0)$ and $P(1)$ are true, and that for any n if $P(n)$ and $P(n+1)$ are true then $P(n+3)$ is true. For what numbers n must $P(n)$ be true?
4. Determine what amounts of postage can be made with 3-cent and 10-cent stamps. Prove your answer.
5. A chocolate bar has n squares arranged in a rectangular pattern. You may break the bar along any horizontal or vertical line separating the squares, and you may do the same with any single piece of the bar once the bar is broken. Show that you need $n - 1$ breaks to separate the bar into n squares no matter how you try to break the bar.

Proofs of the Infinitude of Primes

1. Show that $n! + 1$ must have a prime factor greater than n . Use this to prove that there are infinitely many primes.
2. Modify Euclid's proof to show that there are infinitely many primes of the form $4n + 3$.
3. Use Euclid's proof plus strong induction to show that if p_n is the n -th prime number then $p_n \leq 2^{2^n}$.

More Recursion

1. How many ways can you make 25 cents using only pennies and nickels?
2. Give a recursive formula for $N(c)$, the number of ways to make c cents using only pennies and nickels (You either use at least one nickel or you don't. How many nickels do you use?)
3. Give a recursive formula for $D(c)$, then number of ways to make c cents using pennies, nickels, and dimes.
4. How many ways can you make change for a dollar using pennies, nickels, dimes, and quarters?

You and a friend are playing a game: there is a pile of stones. You take turns removing stones from the pile— during your turn, you may remove 1, 2, or 3 stones. Whoever removes the last stone wins.

1. Prove that the second player has a winning strategy if the pile begins with 8 stones.
2. Prove that the second player has a winning strategy if the pile begins with $4n$ stones, and the first player has a winning strategy in all other cases.