# Chapter 5.2: Strong Induction

Tuesday, July 14

## The Well-Ordering Property

1. Prove that 1 is the smallest natural number (The smallest natural number exists by the W-O.P. Let $a$ be that number, and suppose $a < 1$. Then show that there is a smaller natural number, getting a contradiction).

   Assume $a < 1$ is the smallest (positive) natural number. Then multiplying both sides by $a$ gives $a^2 < a$, which is a contradiction since $a^2$ would be a smaller natural number than $a$.

2. Extend this property to $\mathbb{Z}$: show that if a set $S \subset \mathbb{Z}$ has a lower bound (some $N$ such that $N \leq s$ for all $x \in S$) then $S$ has a least element.

   Suppose $N \leq s$ for all $s \in S \subset \mathbb{Z}$. Then Let $S' = S - N$, and $0 \leq s'$ for all $s' \in S'$. Then $S' \subset \mathbb{N}$, so $S'$ has a smallest element $t$. This implies that $t + N$ is the smallest element in $S$.

3. Find a subset of $\mathbb{Z}$ with no lower bound.

   $\mathbb{Z}$ itself has no lower bound.

4. Show that if a set $S \subset \mathbb{Z}$ or $S \subset \mathbb{N}$ has an upper bound, then $S$ has a greatest element.

   Similar to the earlier question: if $N \geq s$ for all $s \in S$, then let $T = [N - s : s \in S]$. $T$ is a subset of $\mathbb{N}$ and so has a smallest element $t$, meaning that $N - t$ is the largest element of $S$.

## Strong Induction

1. Suppose $P(n)$ is a propositional function, that $P(0)$ and $P(1)$ are true, and that for any $n$ if $P(n)$ and $P(n+1)$ are true then $P(n+2)$ is true. For what numbers $n$ must $P(n)$ be true?

   All $n \geq 0$.

2. Suppose $P(n)$ is a propositional function, that $P(0)$ is true, and that for any $n$ if $P(n)$ is true then $P(n+2)$ and $P(n+3)$ are true. For what numbers $n$ must $P(n)$ be true?

   All $n \geq 2$.

3. Suppose $P(n)$ is a propositional function, that $P(0)$ and $P(1)$ are true, and that for any $n$ if $P(n)$ and $P(n+1)$ are true then $P(n+3)$ is true. For what numbers $n$ must $P(n)$ be true?

   Just $n = 0, n = 1$, and $n = 3$. We do not know that $P(2)$ is true, so cannot conclude that $P(4)$ is true in particular.

4. Determine what amounts of postage can be made with 3-cent and 10-cent stamps. Prove your answer.

   We can make 3, 6, 9, 10, 12, 13, 15, 16, and any number 18 or higher.

   Proof: 18 = 3+3+3+3+3+3, 19 = 10 + 3+3+3, 20 = 10+10 for the base cases.

   For the inductive step, let $n \geq 21$ be some amount of postage. Then $n - 3 \geq 18$, so by inductive assumption (all postages $18 \leq k < n$ can be made with our stamps) $n - 3$ can be made with 3-cent and 10-cent stamps. Then $n$ cents can be made by adding another 3-cent stamp.

5. A chocolate bar has $n$ squares arranged in a rectangular pattern. You may break the bar along any horizontal or vertical line separating the squares, and you may do the same with any single piece of the bar once the bar is broken. Show that you need $n - 1$ breaks to separate the bar into $n$ squares no matter how you try to break the bar.

   Base case: a 1-square bar requires 0 breaks and a 2-square bar requires 1 break.

Inductive step: suppose that a bar of size $k$ requires $k-1$ breaks for all $1 \leq k < n$, and take a bar of size n. Any break will split it into bars of size $a$ and $b$, where $a+b=n$. Each of these in turn requires $(a-1)$ and $(b-1)$ breaks to reduce to squares (by the inductive hypothesis), so the total number of breaks required is $1+(a-1)+(b-1)=(a+b)-1=n-1$.

Thus a 1-square bar requires 0 breaks, and if all smaller bars of size $k<n$ require $(k-1)$ breaks then so does a bar of size $n$. By induction, all bars of size $n$ require $(n-1)$ breaks.

## Proofs of the Infinitude of Primes

1. Show that $n!+1$ must have a prime factor greater than $n$. Use this to prove that there are infinitely many primes.

   $(n!+1)$ must have a prime factor greater than $n$ because (1) it has a prime factorization by the Fundamental Theorem of Arithmetic and (2) since $k|n!$ for all $1 \leq k \leq n$, $\gcd(k, n!+1)=1$ for all $1 \leq k \leq n$.

   Suppose $p$ is the largest prime. Then $p!+1$ has a prime factor greater than $p$, a contradiction.

2. Modify Euclid's proof to show that there are infinitely many primes of the form $4n+3$.

   Suppose $p_1 \dots p_n$ is a list of all of the primes. All of them except for the first (p=2) are odd, and the product of odd numbers is odd. The number $1+\prod p$ is therefore of the form $1+2(2k+1)=4k+3$.

   This new number has a prime factorization. If it is prime, then we have a new prime of the form $4k+3$. Otherwise, it can be factored into other smaller primes. But since the product of numbers equivalent to 1 mod 4 is also equivalent to 1 mod 4, at least one of the new primes in the product must be equivalent to 3 mod 4.

   Either way, we have introduced a prime of the form $4n+3$ not previously on our list. Therefore, there are infinitely many such primes.

3. Use Euclid's proof plus strong induction to show that if $p_n$ is the n-th prime number then $p_n \leq 2^{2^n}$.

   Base case: $p_1 = 2 \leq 2^{2^1}$.

   Inductive step: suppose for $1 \leq k \leq n$ that $p_k \leq 2^{2^n}$. Then since the next prime number is less than or equal to $1+\prod_{k=1}^{n} p_k$, we get

$$
\begin{aligned}
p_{n+1} &\leq 1+\prod_{k=1}^{n} p_k \\
&\leq 1+\prod_{k=1}^{n} 2^{2^k} \\
&\leq 1+2^{\sum_{k=1}^{n} 2^k} \\
&\leq 1+2^{2^{n+1}-1} \\
&\leq 1+2^{2^{n+1}}
\end{aligned}
$$

## More Recursion

1. How many ways can you make 25 cents using only pennies and nickels?

   6: you can use 0,1,2,3,4, or 5 nickels, and use pennies to make up the difference.

2. Give a recursive formula for $N(c)$, the number of ways to make $c$ cents using only pennies and nickels (You either use at least one nickel or you don't. How many nickels do you use?)

$$N(c) = \begin{cases} 1 & 0 \le c \le 4 \\ 1 + N(c-5) & c \ge 5 \end{cases}$$

The cases represent our options as follows: if we have 4 cents or less, we must use pennies and there is only 1 way to do so. If we have $c \ge 5$ cents, we can (1) resolve to use only pennies, or (2) use at least 1 nickel, and then repeat the problem asking how many ways we can make $(c-5)$ cents using nickels and pennies.

3. Give a recursive formula for $D(c)$, then number of ways to make $c$ cents using pennies, nickels, and dimes.

   Similar to the previous logic, we can either stop using dimes and resolve to use only nickels and pennies after point $c$, or use at least one dime and repeat the problem for $(c-10)$ cents.

   $$D(c) = \begin{cases} N(c) & 0 \le c \le 9 \\ N(c) + D(c-10) & c \ge 10 \end{cases}$$

4. How many ways can you make change for a dollar using pennies, nickels, dimes, and quarters?

   The number of ways to make $5c$ cents with nickels and pennies is $c+1$ in general. The number of ways to make $10c$ cents with dimes, nickels, and pennies is $(c+1)^2$ (prove by induction!) and the number of ways to make $10c-5$ cents is $c^2+c$ (also prove by induction!).

   Working recursively with quarters, the number of ways to make change for a dollar is $D(100)+D(75)+D(50)+D(25)+D(0) = 121+72+36+12+1 = 242$ ways to make change for a dollar.

You and a friend are playing a game: there is a pile of stones. You take turns removing stones from the pile— during your turn, you may remove 1, 2, or 3 stones. Whoever removes the last stone wins.

1. Prove that the second player has a winning strategy if the pile begins with 8 stones.

   If the first player takes $s$ stones, then the second player can take $4-s$ stones and leave 4 in the pile. Then no matter how many stones player 1 takes, player 2 can take the rest.

2. Prove that the second player has a winning strategy if the pile begins with 4n stones, and the first player has a winning strategy in all other cases.

   Base case already done. If the pile has $4(n+1)$ stones, then when player 1 takes $s$ stones player 2 can take $(4-s)$ stones, leaving the pile with $4n$ stones—by the inductive hypothesis, a win for player 2. This finishes the proof by induction.

   On the other hand, if the pile has $4n+1, 4n+2$, or $4n+3$ stones, then player 1 can take enough to leave 4n stones, and is now in the same winning position that player 2 was in before.