Chapter 5.1: Induction

Monday, July 13

Fermat's Little Theorem

Evaluate the following:

1. $2^{16} \pmod 5$
2. $3^{32} \pmod 7$
3. $2^{77} \pmod{19}$
4. $2^{18} \pmod{15}$
5. $2^{25} \pmod{21}$
6. $2^{100} \pmod{55}$

(Hard) A composite number n is called a Carmichael number $b^{n-1} \equiv 1 \pmod n$ for every number b such that $\gcd(b, n) = 1$ (their existence is unfortunate, since it means that we cannot use FLT to tell for certain whether a number is prime). Prove: There is one and only one Carmichael number of the form $3 \cdot p \cdot q$, where p and q are prime numbers.

Induction

1. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for $n \geq 0$.
2. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$ for $n \geq 0$.
3. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for $n \geq 1$.
4. Find a closed form for $\sum_{k=1}^{n} (-1)^k k^2$ and prove that it is correct.
5. For what integers is $2^n \geq n^3$ true? Prove it.

From 2 to many

1. Given that $ab = ba$, prove that $a^n b = b^n a$ for all $n \geq 1$.

2. Given that $ab = ba$, prove that $a^n b^m = b^m a^n$ for all $n, m \geq 1$ (let n be arbitrary, then use the previous result and induction on m).

3. Given: if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$. Prove: if $a_i \equiv b_i \pmod{m}$ for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} a_i \equiv \sum_{i=1}^{n} b_i \pmod{m}$.

4. (Calculus) Suppose we know that $\frac{d}{dx}x = 1$ and that for any functions f and g, $(fg)' = f'g + fg'$. Prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \geq 1$.

5. Prove: $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$.

Recursion

1. Define a sequence a_n by $a_0 = 1$, $a_1 = 3$ and $a_n = a_{n-1} + 2 \cdot a_{n-2}$ for $n \geq 2$. Find a_6. Prove that $a_n = \dfrac{2^{n+2} + (-1)^n}{3}$.

2. Define a sequence a_n by $a_0 = 1$, $a_n = 2 \cdot a_{n-1} + 1$ if $n \geq 1$. Find a non-recursive formula for a_n and prove that it is correct.

3. Prove: $\gcd(f_{n+1}, f_n) = 1$ for all $n \geq 0$.

4. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for $n \geq 1$.

5. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ for $n \geq 1$.

6. Show that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ for $n \geq 1$.

7. Prove that $f_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. (Hint: both α and β satisfy the equation $x^2 = x + 1$).

8. Prove that $f_{m+n} = f_{m-1} f_n + f_m f_{n+1}$. (fix n arbitrarily, then use induction on m)

9. Prove (now using induction on n) that $f_m | f_{mn}$ for all $n \geq 1$.

10. Prove that $\gcd(f_m, f_n) = f_{\gcd(m,n)}$.