Chapter 4.4: Systems of Congruences

Friday, July 10

Linear congruences

Find all solutions:

1. $7n \equiv 1 \pmod{19}$

$$\begin{aligned}
19 - 2 \cdot 7 &= 5 \\
7 - 5 &= 2 \\
5 - 2 \cdot 2 &= 1 \\
5 - 2 \cdot (7 - 5) &= 1 \\
3 \cdot 5 - 2 \cdot 7 &= 1 \\
3 \cdot (19 - 2 \cdot 7) - 2 \cdot 7 &= 1 \\
3 \cdot 19 - 8 \cdot 7 &= 1 \\
-8 \cdot 7 &= 1 \pmod{19} \\
11 \cdot 7 &= 1 \pmod{19}
\end{aligned}$$

2. $8n \equiv 3 \pmod{23}$

$$\begin{aligned}
23 - 2 \cdot 8 &= 7 \\
8 - 7 &= 1 \\
8 - (23 - 2 \cdot 8) &= 1 \\
3 \cdot 8 - 23 &= 1 \\
3 \cdot 8 &\equiv 1 \pmod{23} \\
9 \cdot 8 &\equiv 3 \pmod{23}
\end{aligned}$$

3. $5n \equiv 6 \pmod{11}$

   Try this with some trial and error:

$$\begin{aligned}
5 \cdot 2 &\equiv 10 \pmod{11} \\
5 \cdot 2 &\equiv -1 \pmod{11} \\
5 \cdot 10 &\equiv -5 \pmod{11} \\
5 \cdot 10 &\equiv -6 \pmod{11}
\end{aligned}$$

4. $7n \equiv 4 \pmod{19}$

   From before: $11 \cdot 7 \equiv 1 \pmod{19}$, so $44 \cdot 7 \equiv 4 \pmod{19}$. Then $44 \bmod 19 = 6$, so $6 \cdot 7 \equiv 4 \pmod{19}$.

5. $19n \equiv 1 \pmod{7}$

   From before: $3 \cdot 19 - 8 \cdot 7 = 1$, so $19 \cdot 3 \equiv 1 \pmod{7}$.

6. $8n \equiv 8 \pmod{31}$

   31 is prime, so $8n \equiv 8 \pmod{31} \Leftrightarrow n \equiv 1 \pmod{31}$.

7. $8n \equiv 18 \pmod{24}$

   8 and 24 are both divisible by 8 but 18 is not. The system has no solutions.

8. $7n \equiv 18 \pmod{35}$

   7 and 35 are both divisible by 7 but 18 is not. No solutions.

9. $7n \equiv 21 \pmod{35}$

   All numbers are divisible by 7, so divide by 7 all around to get $n \equiv 3 \pmod 5$. Mod 35, the solutions are $n = 3, 8, 13, 18, 23, 28, 33$.

10. $3n \equiv 9 \pmod{15}$

    Divide by 3 to get $n \equiv 3 \pmod 5$. mod 15, the solutions are $n = 3, 8, 13$.

11. $15n \equiv 13 \pmod{25}$

    15 and 25 are divisible by 5 but 13 is not. No solutions.

12. $15n \equiv 20 \pmod{25}$

    Divide by 5 to get $3n \equiv 4 \pmod 5$, which has the solution $n \equiv 3 \pmod 5$. Mod 25, the solutions are $n = 3, 8, 13, 18, 23$.

Chinese Remainder Theorem

Decide whether the system has a solution. If it does, find it.

1. $x \equiv 3 \pmod 8$, $x \equiv 1 \pmod 7$

   Try $x = 8a+7b$. mod 8, we get $3 \equiv x \equiv 7b \pmod 8$, and solving gives $b = 5$. mod 7, we get $1 \equiv x \equiv 8a \pmod 7$, so $a \equiv 1 \pmod 7$. Therefore one solution is $x = 8 + 7 \cdot 5 = 43$.

2. $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod{13}$

   Try $x = 5a + 13b$. mod 5, we get $2 \equiv x \equiv 13b \equiv 3b \pmod 5$, so $b = 4$ is a solution. mod 13, we get $3 \equiv x \equiv 5a \pmod{13}$ with $a = 11$ as a solution. Therefore one solution is $11 \cdot 5 + 4 \cdot 13 = 107$, which is equivalent to $42 \pmod{65}$.

3. $x \equiv 7 \pmod 6$, $x \equiv 4 \pmod 8$

   The first equation suggests that x is odd but the second requires x to be even. No solutions.

4. $x \equiv 1 \pmod 6$, $x \equiv 5 \pmod 8$

   Since $\gcd(6, 8) = 2$ but both equations give $x \equiv 1 \pmod 2$, the equations are compatible. x must be odd, so say $x = 2k + 1$. This leads to the equations $2k \equiv 0 \pmod 6$ and $2k \equiv 4 \pmod 8$, and dividing by 2 gives $k \equiv 0 \pmod 3$ and $k \equiv 2 \pmod 4$, with the solution $k = 6$. Thus $x = 2k+1 = 2 \cdot 6+1 = 13$ is a solution (and the only solution mod 24).

5. $x \equiv 8 \pmod{15}$, $x \equiv 3 \pmod{10}$, $x \equiv 1 \pmod 6$

   The first equation implies $x \equiv 2 \pmod 3$ but the second requires that $x \equiv 1 \pmod 3$.

6. $x \equiv 2 \pmod 3$, $x \equiv 5 \pmod 7$, $x \equiv 3 \pmod{11}$

   Try a solution of the form $x = 3 \cdot 7 \cdot a + 3 \cdot 11 \cdot b + 7 \cdot 11 \cdot c$. Taking the remainders mod 3, 7, and 11 in turn gives the three equations $2 \equiv 77c \equiv 2c \pmod 3$ (so c = 1), $5 \equiv 33 \cdot b \equiv 5 \cdot b \pmod 7$ (so b = 1), and $3 \equiv 21 \cdot a \equiv -a \pmod{11}$ (so a = -3).

   One solution is therefore $x = -3 \cdot 21 + 1 \cdot 33 + 1 \cdot 77 = 47$. This solution is also unique mod $3 \cdot 7 \cdot 11 = 231$.

Decide whether the system has a solution (and if it does, find all solutions) by solving the system for each prime factor separately.

1. $n^2 \equiv 11 \pmod{35}$

   Working over each prime factor separately gives $n^2 \equiv 1 \pmod 5$ and $n^2 \equiv 4 \pmod 7$, so $n = \pm 1 \pmod 5$ and $n = \pm 2 \pmod 7$.

   Finding all solutions using the Chinese Remainder Theorem would be a real pain, so we'll go by brute force: look at all the numbers that are $\pm 2 \pmod 7$ and see which ones are also ± 1 mod 5 (that is, end in a 1, 4, 6, or 9):

   The options (mod 35) are $n = 2, 5, 9, 12, 16, 19, 23, 26, 30, 33$. Of these, the ones that work mod 5 are 9, 16, 19, and 26.

2. $n^2 \equiv 12 \pmod{15}$

   Get the equations $n^2 \equiv 0 \pmod 3$ and $n^2 \equiv 2 \pmod 5$... the second equation has no solutions, so there are no solutions to $n^2 \equiv 12 \pmod{15}$.

3. $n^2 \equiv 15 \pmod{77}$

   Get the equations $n^2 \equiv 1 \pmod 7$ and $n^2 \equiv 4 \pmod{11}$, so $n = \pm 1$ mod 7 and $n = \pm 2$ mod 11. Look at the ones that work mod 11 and then filter out to see which work for 7:

   The options are n = 2, 9, 13, 20, 24, 31, 35, 42, 46, 53, 57, 64, 68, 75. Of these, 13, 20, 57, and 64 are ± 1 mod 7. These are the four solutions.

   Note: $13 + 64 = 20 + 57 = 77$, so these solutions again come in pairs. (That is, if $n^2 \equiv 15 \pmod{77}$ then $(-n)^2 \equiv 15 \pmod{77}$.)

4. $n^2 \equiv 5 \pmod{33}$

   This leads to the equation $n^2 \equiv 2 \pmod 3$, which has no solutions.

Show that if p and q are primes with $p, q > 2$. then $n^2 \equiv 1 \pmod{pq}$ has four distinct solutions.
Use the Chinese Remainder Theorem on $n \equiv \pm 1 \pmod p$, $n \equiv \pm 1 \pmod q$.

Fermat's Little Theorem

Evaluate:

1. $5^{100} \pmod 7$

   $5^6 \equiv 1 \pmod 7$, so $5^{100} \equiv 5^4 \equiv (-2)^4 \equiv 16 \equiv 2 \pmod 7$.

2. $3^{32} \pmod 5$

   $3^4 \equiv 1 \pmod 5$ so $3^{32} \equiv 1 \pmod 5$.

3. $17^{73} \pmod{19}$

   $17^{18} \equiv 1 \pmod{19}$, so $17^{73} \equiv 17 \pmod{19}$.

4. $8^{32} \pmod{35}$

   We cannot use Fermat's Little Theorem directly, but we can solve mod 5 and mod 7 separately. $8^4 \equiv 1 \pmod{5}$, so $8^{32} \equiv 1 \pmod{5}$. Then $8 \equiv 1 \pmod{7}$ so $8^{32} \equiv 1 \pmod{7}$.

   If $x \equiv 1 \pmod{5}$ and $x \equiv 1 \pmod{7}$ then $x \equiv 1 \pmod{35}$ (1 is a solution mod 35, and by CRT is the unique solution). Therefore $8^{32} \equiv 1 \pmod{35}$

5. $8^{20} \pmod{15}$

   $8 \equiv (-1) \pmod{3}$ so $8^{20} \equiv 1 \pmod{3}$. $8^4 \equiv 1 \pmod{5}$ so $8^{20} \equiv 1 \pmod{5}$. Putting the two together, $8^{20} \equiv 1 \pmod{15}$.

6. $15^{37} \pmod{21}$

   15 is divisible by 3 so $15^{37} \equiv 0 \pmod{3}$. 15 is 1 mod 7 so $15^{37} \equiv 1 \pmod{7}$. Therefore $15^{37} \equiv 15 \pmod{21}$.

Show that $n^2 \equiv -1 \pmod{103}$ has no solutions.
FLT says that if $n \not\equiv 0 \pmod{103}$ then $n^{102} \equiv 1$ (mod 1)03. But if $n^2 \equiv -1$ then $n^4 \equiv 1$, so $n^{100} \equiv 1 \pmod{103}$. So $n^{100} \equiv n^{102} \pmod{103}$ and so $n^2 \equiv 1 \pmod{103}$. Therefore there are no solutions to $n^2 \equiv -1 \pmod{103}$.

Use Fermat's Little Theorem with base $n = 2$ to prove that 9 is not prime.
$2^8 \equiv 4^4 \equiv 16^2 \equiv (-2)^2 \equiv 4 \not\equiv 1 \pmod{9}$.

Use Wilson's Theorem to show that 7 is prime.
$6! = 120 = 119 + 1 = 17 \cdot 7 + 1 \equiv 1 \pmod{7}$, so 7 is prime.