Chapters 2.4, 4.1

Tuesday, July 7

Sums

What is a closed form for $\sum_{k=1}^{n}(-1)^k k^2$? Find the first few values, then come up with a conjecture.

$$S_1 = -1, S_2 = 3, S_3 = -6, S_4 = 10, S_5 = 15\dots$$

A reasonable guess would be $S_n = (-1)^n \cdot n(n+1)/2$. The formal proof will have to wait until we get to induction.

Find (at least) two ways to express the sum of the triangle below:

1					
2	1				
3	3	1			
4	5	4	1		
5	7	7	5	1	
6	9	10	9	6	1

Method 1: Sum by columns, then by rows. If we index the columns from 1 to 6, and we label the top number in each column by $r = 0$, then the columns can be written in the following pattern in terms of r:

$$\begin{aligned} 1, 2, 3, 4, 5, 6 \dots &= 1 + r \\ 1, 3, 5, 7, 9 \dots &= 1 + 2r \\ 1, 4, 7, 10 \dots &= 1 + 3r \\ &\dots \end{aligned}$$

The pattern changes with each column c, but we can fit the numbers to the pattern $S(c, r) = 1 + cr$. This makes the double sum $\displaystyle\sum_{c=1}^{6}\sum_{r=0}^{6-c} 1 + cr$.

Method 2: Sum by diagonals, then by columns: label the diagonals from $d = 0$ to 5. Then label the columns from $c = 1$ to 6. Then we can find the following patterns in terms of d and c:

$$\begin{aligned} 1, 1, 1, 1, 1, 1 &= 1 \\ 2, 3, 4, 5, 6 &= 1 + c \\ 3, 5, 7, 9 &= 1 + 2c \\ 4, 7, 10 &= 1 + 3c \end{aligned}$$

This fits the more general pattern $S(d, c) = 1 + dc$. The double sum then becomes $\displaystyle\sum_{d=0}^{5}\sum_{c=1}^{6-d} 1 + dc$.

Put in summation notation: what is the x^2 coefficient of $(a_1x + a_0)(b_1x + b_0)(c_1x + c_0)$? What is the x^2 coefficient of $(x + 1)^3$?

The x^2 coefficient of the product is $(a_1b_1c_0 + a_1b_0c_1 + a_0b_1c_1)$. In summation notation this would be

$$\sum_{\substack{0 \le i,j,k \le 1 \\ i+j+k=2}} a_i b_j c_k$$

Here is a board, with some X's on it:

			X				
		X				X	
X	X						
							X
						X	
			X	X			
					X		
	X					X	

Each square is worth some number of points: for every X it shares a row with (including the square it is on), it gets a point. For every X it shares a column with (including the square it is on), it *loses* a point. What is the sum of the point values of all the squares on the board? (Hint: you have to sum over the right variable. . .)

This would be hard if we tried to sum first over the rows or the columns; instead count the contribution each letter "X" makes to the total score. Each X gives 8 points to the squares in its row and takes 8 points away from the squares in its column, so the net contribution is zero. Thus the sum of the scores of all the squares is 0.

Divisibility

True or false? If true, prove. If false, find a counterexample.

1. If $a|b$ and $b|c$ then $a|c$.

   True: if $a|b$ and $b|c$ then there exist $k, m \in \mathbb{Z}$ such that $ak = b$ and $bm = c$. Then $a(km) = (ak)m = bm = c$, and so $a|c$.

2. If $a|b$ and $a|c$ then $a|b$.

   Trivially true.

3. If $a|b$ and $b|a$ then $a = b$.

   False: $a = 3, b = -3$ is a counterexample. It would be true to instead say $a = \pm b$ or $|a| = |b|$.

4. If $a|c$ and $b|c$ then either $a|b$ or $b|a$.

   False: $a = 12, b = 8, c = 24$. Or $a = 3, b = 5, c = 15$.

5. If $a|b$ and $a|c$ then $a|(mb + nc)$ for any $m, n \in \mathbb{Z}$.

   True: Since $a|b$, $a|mb$ for any $m \in \mathbb{Z}$. Since $a|c$, $a|nc$ for any $n \in \mathbb{Z}$. Then since $a|mb$ and $a|nc$, it follows that $a|(mb + nc)$ for any $m, n \in \mathbb{Z}$.

   Application: If we have 4-cent and $6-$cent stamps, we cannot combine them to make 37 cents because 4 and 6 are both even, so any combination of stamps must also have an even price.

6. $a|a$ for any a.

   True: $a \cdot 1 = a$.

7. $a|0$ for any a.

   True: $a \cdot 0 = 0$.

8. $1|a$ for any a.

   True: $1 \cdot a = a$ for any a.

9. $0|a$ for any a.

   False: For any k, $0 \cdot k = 0$, so there does not exist any k such that $0 \cdot k = 1$ (or any other non-zero number, for that matter).

10. $0|a$ if and only if $a = 0$.

    True: The previous problem showed that $0 \nmid a$ if $a \neq 0$, so we just have to show that $0|0$, which is true because $0 \cdot 1 = 0$.

11. Suppose $a|b$. Then $a|(b + c)$ if and only if $a|c$.

    True: We know that if $a|b$ and $a|c$ then $a|(b + c)$ from the book. Conversely, if $a|b$ then $a|(-b)$, so if $a|(b + c)$ then $a|(b + c) + (-b)$, meaning $a|c$.

    This case shows a way to get new rules from previous rules, but we could also do this directly: if $a|b$ then $ak = b$ for some k. If $a|(b + c)$ then $aj = b + c$ for some j. This means that $a(j - k) = aj - ak = b + c - b = c$, and so $a|c$.

12. If $2|n$ and $4|n$ then $8|n$.

    False: $n = \pm 4$ are the only two counterexamples.

13. If $2|n$ and $3|n$ then $6|n$.

    True: If $2|n$ then $n = 2k$ for some k. If $3|n$ then $n = 3j$ for some j. Thererfore $2k = 3j$ for some $j, k \in \mathbb{Z}$. Since the left hand side is even, the right hand side must be even as well. Thus j must be even, so $j = 2m$ for some m. Therefore $n = 3j = 3(2m) = 6m$, and so $6|n$.

14. How can you tell when a number is divisible by $\{2, 3, 4, 5, 6, 7, 8, 9, 11, 17\}$?

    (a) A number is divisible by 2/5 if and only if the last digit is divisible by 2/5.

    (b) A number is divisible by 3/9 if and only if the sum of its digits is divisible by 3/9.

    (c) A number is divisible by 4/8 if and only if the last 2/3 digits are divisible by 4/8.

    (d) A number is divisible by 6 if and only if it is divisible by 2 and 3.

    (e) A number is divisible by 11 if and only if the sum of its even-place digits minus the sum of its odd-place digits (or vice versa) is divisible by 11.

    This proof uses the fact that 99 is divisible by 11 (and therefore so is 999...9, for any even number of 9's). So, for example,

$$\begin{aligned} 11|23456 &\Leftrightarrow 11|2 \cdot 10000 + 34 \cdot 100 + 56 \\ &\Leftrightarrow 11|2 \cdot 9999 + 234 \cdot 99 + 34 + 56 \\ &\Leftrightarrow 11|2 + 34 + 56 \\ &\Leftrightarrow 11|(3+5) \cdot 10 + (2+4+6) \\ &\Leftrightarrow 11|(3+5) \cdot 10 - (3+5) \cdot 11 + (2+4+6) \\ &\Leftrightarrow 11|(2+4+6) - (3+5) \end{aligned}$$

    (f) A similar trick to tell whether a number is divisible by 7:

$$\begin{aligned} 7|10t + u &\Leftrightarrow 7|10t + u - 21u \\ &\Leftrightarrow 7|10t - 20u \\ &\Leftrightarrow 7|10(t - 2u) \\ &\Leftrightarrow 7|t - 2u \end{aligned}$$

    So for example, 3444 is divisible by 7 if and only if $344 - 2 \cdot 4 = 336$ is divisible by if and only if $33 - 2 \cdot 6 = 21$ is divisible by 7 (which it is).

    Note: We have not yet justified the final step in this proof–that $7|10(t-2u)$ if and only if $7|(t-2u)$. We will do so soon.

    (g) $17|10t + u$ if and only if $17|t - 5u$. Proof similar to the proof for divisibility by 7.