Chapters 2.4, 3.2

Tuesday, June 30

Sum and Product Notation

Evaluate:

1. $\sum_{i=1}^{4} i^2 + 3i = 4 + 10 + 18 + 28 = 60$

2. $\sum_{j=15}^{30} j^2 = \frac{30 \cdot 31 \cdot 61}{6} - \frac{14 \cdot 15 \cdot 29}{6} = 8440$

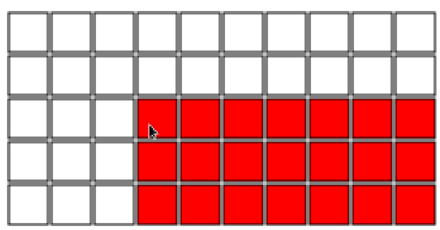
3. $\sum_{i=1}^{3} \sum_{j=1}^{3} i + j + 1 = 3 + 4 + 5 + 4 + 5 + 6 + 5 + 6 + 7 = 12 + 15 + 18 = 45$

4. $\prod_{k=1}^{3} \sum_{i=1}^{4} ik = \prod_{k=1}^{3} k(\sum_{i=1}^{4} i) = 10 \prod_{k=1}^{3} k = 10 \cdot 6 = 60$

Show using the sum formulas that the sum of the first n positive odd numbers is equal to n^2.

$$1 + 3 + 5 + \ldots + (2n-1) = \sum_{i=1}^{n} 2i - 1 = -n + 2\sum_{i=1}^{n} i = -n + \frac{1}{2}\frac{n^2+n}{2} = -n + n^2 + n = n^2$$

How many rectangles of any size and orientation can be found on this 5×10 grid? (Note that perfect squares count as rectangles.)
Using the formula below gives the number of rectangles as $5 \cdot 10 \cdot 6 \cdot 11/4 = 825$.



How many rectangles on an $m \times n$ grid?
In general there are $(m+1-i)(n+1-j)$ ways to place an $(i \times j)$ rectangle on the grid. The number of rectangles on an $m \times n$ grid is therefore

$$\begin{aligned}
\sum_{i=1}^{m} \sum_{j=1}^{n} (m+1-i)(n+1-j) &= \left(\sum_{i=1}^{m} m+1-i\right)\left(\sum_{j=1}^{n} n+1-j\right) \\
&= \left(\sum_{i=1}^{m} i\right)\left(\sum_{j=1}^{n} j\right) \\
&= \frac{mn(m+1)(n+1)}{4}
\end{aligned}$$

Find a formula (using summation notation) for the sum of the numbers on each row of this triangle. Bonus: show that the sum of each row is a cube.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 5 & & \\ & 7 & & 9 & & 11 & \\ 13 & & 15 & & 17 & & 19 \\ & \vdots & & \vdots & & \vdots & \end{array}$$

First: Define T_n as the n-th *triangular number*: $T_n = \sum_{i=1}^{n} i = \frac{n^2+n}{2}$.
Then, the sum of the numbers in a single row is equal to the sum of the numbers in the triangle up to that point minus the sum of the numbers in the previous triangle. Using the fact that $1+3+5+\ldots+(2n-1) = n^2$ shows that the sum of the numbers in the first n rows combined is $(T_n)^2$ (in other words, $1 = 1^2, 1+3+5 = 3^2, 1+3+\ldots+11 = 6^2$, then $10^2, 15^2$, etc.). The sum of the numbers in any single row of numbers is therefore $(T_n)^2 - (T_{n-1})^2$. Using the formula for triangular numbers, this becomes

$$T_n^2 - (T_{n-1})^2 = \frac{n^2(n+1)^2}{4} - \frac{n^2(n-1)^2}{4} = \frac{n^2}{4}(n^2+2n+1-n^2+2n-1) = \frac{n^2}{4}(4n) = n^3$$

Big-Oh Notation

Show (without finding an exact formula for the sum) that $\sum_{i=1}^{n} i^5$ is $O(n^6)$.
For any n, $\sum_{i=1}^{n} i^5 \leq \sum_{i=1}^{n} n^5 n \cdot n^5 = n^6$. Thus $\sum_{i=1}^{n} i^5$ is $O(n^6)$.
Define $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. Prove: if $a_n \sim b_n$ and $b_n \sim c_n$ then $a_n \sim c_n$.
$\lim_{n\to\infty} a_n/c_n = \lim_{n\to\infty} a_n/b_n \cdot b_n/c_n = 1 \cdot 1 = 1$.
Suppose the sequence a_n is $\Theta(n^k)$. If we define $b_n = \sum_{i=1}^{n} a_i$, then prove that b_n is $\Theta(n^{k+1})$. What does this say about building 2-dimensional and 3-dimensional objects?
Proof omitted. Long story short: if we build a 2-D object as a sum of 1-D objects (which have size $\Theta(n)$), then any such proportionally growing 2-D object will have size $\Theta(n^2)$ and proportionally growing 3-D object will have size $\Theta(n^3)$.
Prove that if $a_n = O(b_n)$ and $c_n = O(b_n)$ then $a_n + c_n = O(b_n)$.
There is some pair (k, C) such that if $n > k$ then $|a_n| < C|b_n|$, ditto with a pair (k', C') for c_n. Let $l = \max(k, k')$. Then if $n > l$, we have $|a_n + c_n| \leq |a_n| + |c_n| < C|b_n| + C'|b_n| = (C + C')|b_n|$. Thus $a_n + c_n = O(b_n)$.

Disprove: if $a_n = O(b_n)$ and $a_n = O(c_n)$ then $a_n = O(b_n + c_n)$ (think negative numbers.)
$a_n = b_n = n$, $c_n = -n$.

Let $a_n, b_n > 0$. Prove: $a_n = \Theta(b_n) \Leftrightarrow \ln(a_n) = \ln(b_n) + O(1)$. (What does $O(1)$ mean?)