Chapters 1.5,2.3: Quantifiers and Functions

Tuesday, June 30

Nested Quantifiers

Play the word-making games with three letters: let W be the set of all three-letter tuples $(x, y, z) \in A^3$ that make valid English words. Player 1 names a letter, then Player 2, then Player 1 again. Decide who has the winning strategy if. . .

1. Player 2 wins when a valid English word is made.
2. Player 1 wins when a valid English word is made.

Re-write in English, and prove or disprove. The domain is $\mathbb{R}$ unless stated otherwise. Write the negation of the statement in quantifier notation. If you so desire, play a few rounds of the Quantifier Game with a partner.

1. $(\forall x)(\exists y)(y^2 < x)$
2. $(\exists x)(\forall y)(y^2 > x)$
3. $(\forall x)(\exists y)(|x - y| \in \mathbb{Z})$
4. $(\exists x)(\forall y)(\exists z)(yz = x)$
5. $(\exists x)(\forall y)(\exists z)(x + y = z)$
6. $(\exists x)(\forall y)(\forall z)(x + y = z)$

Consider the conjecture "For any even number n there are primes x and y such that $x + y = n$." Which of the following, if any, serves as a counterexample?

1. 20 is even [EDIT: originally said "prime"] and $20 = 5 + 15$, but 15 is not prime.
2. 2 is prime and 13 is prime, but $13 + 2 = 15$, which is not even.
3. 2 is even, but the sum of any pair of primes is at least 4.

Write in quantifier notation, and prove:

1. 5 is not the maximum of the interval $(3, 6)$.
2. For any real numbers $s < r$, the interval (s, r) has no maximum.

Functions

1. Find a function $f : \mathbb{R} \to \mathbb{Z}$ that is onto.
2. Find a function $f : \mathbb{Z} \to \mathbb{R}$ that is one-to-one.
3. Find a function $f : \mathbb{Z} \to \mathbb{Z}$ that is a bijection but is not the identity function.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

1. What is $f([-1, 3])$?
2. Prove that f is not one-to-one (make sure to say what this means in quantifier notation!)
3. Prove that f is not onto.
4. If we change the domain of the function to $[r \in \mathbb{R} | r \geq 0]$, how do the previous two answers change?
5. What if we change the target space [EDIT: originally said "range"] to $[r \in \mathbb{R} | r \geq 0]$? What if both the domain *and* target space are $[r \in \mathbb{R} | r \geq 0]$?

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$, and define $g : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = 2x$. Prove that f is a bijection but g is not.

Say that $A, B \subset X$ and f is a function from X to C. Prove that if $A \subset B$ then $f(A) \subset f(B)$. Find a counterexample to show that the converse is false.

Prove that if $g : A \to B$ and $f : B \to C$ are both onto, then $f \circ g : A \to C$ is also onto. Do the same for "one-to-one." [EDIT: originally had the order of f and g switched, which would not make $f \circ g$ a valid function]