Chapters 1.5,2.3: Quantifiers and Functions

Tuesday, June 30

Nested Quantifiers

Play the word-making games with three letters: let W be the set of all three-letter tuples $(x, y, z) \in A^3$ that make valid English words. Player 1 names a letter, then Player 2, then Player 1 again. Decide who has the winning strategy if...

1. Player 2 wins when a valid English word is made.

   Player 1 has the winning strategy, especially since they can make the final letter "q." In quantifier notation, $(\exists x)(\forall y)(\exists z)((x, y, z) \notin W)$.

2. Player 1 wins when a valid English word is made.

   Player 2 can probably win this... making "j" the middle letter will probably prevent Player 1 from making a valid word. In quantifier notation, $(\forall x)(\exists y)(\forall z)((x, y, z) \notin W)$. No matter what player 1 does, player 2 has a move that will win (regardless of player 1's final move).

Re-write in English, and prove or disprove. The domain is $\mathbb{R}$ unless stated otherwise. Write the negation of the statement in quantifier notation. If you so desire, play a few rounds of the Quantifier Game with a partner.

1. $(\forall x)(\exists y)(y^2 < x)$

   For every x there is a y such that $y^2 < x$. This is false since if $x = 0$, there is no $y \in \mathbb{R}$ such that $y^2 < 0$. In quantifier notation, the negation becomes $(\exists x)(\forall y)(y^2 \geq x)$.

2. $(\exists x)(\forall y)(y^2 > x)$

   There is an x such that $y^2 > x$ for all y. This is true, and $x = -1$ serves as an example choice for x.

   The negation would be $(\forall x)(\exists y)(y^2 \leq x)$.

3. $(\forall x)(\exists y)(|x - y| \in \mathbb{Z})$

   For every x there is a y such that $|x - y|$ is an integer (which is really the same thing as saying that $x - y$ is an integer, since $|x| \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}$). This statement is true, since for any x we can set $y = x$, so that $x - y = 0$.

   The negation would be $(\exists x)(\forall y)(|x - y| \notin \mathbb{Z})$.

4. $(\exists x)(\forall y)(\exists z)(yz = x)$

   There is some x such that for any y we can find a z so that $yz = x$. This is true, and the only value for x that works is $x = 0$ (if $x \neq 0$, then $y = 0$ would be a counterexample to the claim).

   The negation is $(\forall x)(\exists y)(\forall z)(yz \neq x)$.

5. $(\exists x)(\forall y)(\exists z)(x + y = z)$

   There is some x such that for any y we can find a z so that $x + y = z$. This is true, and any value of x will work since after y is picked we can just set $z = x + y$. (This means that the statement can be strengthened to $(\forall x)(\forall y)(\exists z)(x + y = z)$).

   The negation would be $(\forall x)(\exists y)(\forall z)(x + y \neq z)$.

6. $(\exists x)(\forall y)(\forall z)(x + y = z)$

   There is some x such that $x + y = z$ for any y and z. This claim is quite false, since for any x we can pick $y = 0$ and $z = x + 1$, secure in the knowledge that $x + 0 = x + 1$ is always false.

   The negation is $(\forall x)(\exists y, z)(x + y \neq z)$.

Consider the conjecture "For any even number n there are primes x and y such that $x + y = n$." Which of the following, if any, serves as a counterexample?

1. 20 is even [EDIT: originally said "prime"] and 20 = 5 + 15, but 15 is not prime.
2. 2 is prime and 13 is prime, but 13 + 2 = 15, which is not even.
3. 2 is even, but the sum of any pair of primes is at least 4.

In quantifier notation, this statement would be $(\forall n \in 2\mathbb{Z})(\exists x, y \in P)(x + y = n)$, where $2\mathbb{Z}$ is the set of even numbers and P is the set of primes.
Put most naturally in English, the negation is "There is some even integer that cannot be written as the sum of two prime numbers." In quantifier notation this would look like $(\exists n \in 2\mathbb{Z})(\nexists x, y \in P)(x + y = n)$. If we carry the negation all the way through, this becomes $(\exists n \in 2\mathbb{Z})(\forall x, y \in P)(x + y \neq n)$, although this makes a little less sense in English.
So our counterexample looks like an even number that *cannot* be written as the sum of two primes. (2) looks nothing like a counterexample.
(1) is not a counterexample since we did not say that ANY pair (x, y) with $x+y = n$ needed both x and y to be prime, we conjectured that some such pair EXISTED. In this case, there are two pairs of prime numbers that sum to 20: $3 + 17$ and $7 + 13$.
(3) is a counterexample, since 2 cannot be written as the sum of two prime numbers (1 is not prime by convention).
The proper conjecture (Goldbach's Conjecture) is "Any even number *greater than 2* can be written as the sum of two primes." We do not know whether this conjecture is true or false.
Write in quantifier notation, and prove:

1. 5 is not the maximum of the interval $(3, 6)$.

   The statement "5 is the maximum" would look like $(5 \in (3, 6)) \wedge (\forall x \in (3, 6))(x \leq 5)$. We know that 5 is in the interval, so we only need to bother disproving the second part of that statement.

   The negation is $(\exists x \in (3, 6))(x > 5)$— there is some number in the interval greater than 5. Since 5.5 is such a number, we can see that 5 is not the maximum.

2. For any real numbers $s < r$, the interval (s, r) has no maximum.

   $(\forall s < r \in \mathbb{R})(\nexists x \in (s, r))(\forall y \in (s, r))(y \leq x)$ would be the most natural way to put this in quantifier notation. Carrying the negation all the way through makes it $(\forall s < r \in \mathbb{R})(\forall x \in (s, r))(\exists y \in (s, r))(y > x)$.

   Proof: Let (s, r) be an arbitrary interval in $\mathbb{R}$ with $s < r$, and let x be an arbitrary number in (s, r) (so that $s < x < r$). Then $s < x < \frac{x+r}{2} < r$, so $\frac{x+r}{2}$ is a number in (s, r) that is greater than x, proving that x is not the maximum.

   Since we showed that x is not the maximum for an arbitrary x, this proves that the maximum of (s, r) does not exist.

Functions

1. Find a function $f : \mathbb{R} \to \mathbb{Z}$ that is onto.

   The floor and ceiling functions work here: $f(x) = \lfloor x \rfloor$ or $f(x) = \lceil x \rceil$.

2. Find a function $f : \mathbb{Z} \to \mathbb{R}$ that is one-to-one.

   The function $f(x) = x$ will do.

3. Find a function $f : \mathbb{Z} \to \mathbb{Z}$ that is a bijection but is not the identity function.

   $f(x) = -x$ works.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

1. What is $f([-1, 3])$?

$$f([-1, 3]) = [0, 9]$$

2. Prove that f is not one-to-one (make sure to say what this means in quantifier notation!)

   $(\exists x, y)(x \neq y \wedge f(x) = f(y))$. The pair $x = -2, y = 2$ will do since $x \neq y$ but $(-2)^2 = 4 = 2^2$, so $f(x) = f(y)$.

3. Prove that f is not onto.

   The quantifier notation would be $(\exists x)(\forall y)(f(y) \neq x)$, or "There is some x such that $f(y) \neq x$ for any y."

   Proof: Since $x^2 \geq 0$ for all x, there is no number $x \in \mathbb{R}$ such that $f(x) = x^2 = -1$.

4. If we change the domain of the function to $[r \in \mathbb{R} | r \geq 0]$, how do the previous two answers change?

   The function would be 1-1 but not onto.

5. What if we change the target space [EDIT: originally said "range"] to $[r \in \mathbb{R} | r \geq 0]$? What if both the domain *and* target space are $[r \in \mathbb{R} | r \geq 0]$?

   If just the target is changed, then the function would be onto but not 1-1. If both the source and target are changed to the set of non-negative reals, then the function would be 1-1 and onto — a bijection. (Think of this as paying attention to only the top right quadrant of the graph for $y = x^2$).

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$, and define $g : \mathbb{Z} \to \mathbb{Z}$ by $g(x) = 2x$. Prove that f is a bijection but g is not.
Proof that f is 1-1: If $f(y) = f(x)$ then $2y = 2x$, so $y = x$.
Proof that f is onto: for any $y \in \mathbb{R}$, $f(\frac{y}{2}) = 2\frac{y}{2} = y$. Thus f is a bijection.

Proof that g is not onto: $2n$ is by definition even for any n, so $g(n)$ is even for all n. Therefore there is no n such that $g(n) = 1$ (or any odd integer), and so g is not onto.

Say that $A, B \subset X$ and f is a function from X to C. Prove that if $A \subset B$ then $f(A) \subset f(B)$. Find a counterexample to show that the converse is false.
We want to prove that if $x \in f(A)$ then $x \in f(B)$.

Proof: Let $x \in f(A) \subset X$. This means that there is some $y \in A$ such that $f(y) = x$. But since $y \in A$ and $A \subset B$, it follows that $y \in B$. Therefore, there is some $y \in B$ such that $f(y) = x$, meaning that $x \in f(B)$.

As a counterexample, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for all $x \in \mathbb{R}$. Let $A = \{1\}$ and let $B = \{2\}$. Then $f(A) = f(B) = \{0\}$ but $A \not\subseteq B$.

Prove that if $g : A \to B$ and $f : B \to C$ are both onto, then $f \circ g : A \to C$ is also onto. Do the same for "one-to-one." [EDIT: originally had the order of f and g switched, which would not make $f \circ g$ a valid function]

Proof: Take an arbitrary $c \in C$. Since f is onto there is some $b \in B$ such that $f(b) = c$. Since g is onto there is some $a \in A$ such that $g(a) = b$. It follows that $f \circ g(a) = f(g(a)) = f(b) = c$, and so there exists $a \in A$ such that $f \circ g(a) = c$. Therefore, $f \circ g$ is onto.

Proof for "one-to-one": Let $f(g(x)) = f(g(y))$. Since f is 1-1, this implies that $g(x) = g(y)$. Since g is 1-1, this implies that $x = y$. Therefore $f \circ g$ is 1-1.