Chapters 1.4-1.5: Quantifiers
Monday, Juney 29

English to Quantifier

1. There is a unique number x such that $x^2 = 0$.

2. 1 is the smallest positive integer.

3. Every integer is either odd or even.

4. Either all integers or odd, or all integers are even.

5. Any number divisible by 4 is also divisible by 2.

6. The sum of any two even numbers is even.

7. For any $n \geq 2$ there is a prime number between n and $2n$.

8. Every even number greater than 2 can be written as the sum of two primes.

9. Everybody doesn't like something, but nobody doesn't like Sarah Lee. (2 propositions)

10. Everybody loves my baby, but my baby don't love nobody but me. (2 propositions, maybe 3)

11. The previous lyrics are from an old song popularized by Louis Armstrong. Prove: if we take "everybody" literally, then "my baby" and "me" must actually be the same person!

Quantifier to English

Write each of the following statements in English and decide whether they are true or false. The domain is $\mathbb{R}$ unless otherwise specified.

1. $(\forall x)(x^2 > 0)$

2. $(\exists x)(\forall y)(x > y)$

3. $(\forall x)(\exists y)(x > y)$

4. $(\forall x > 0)(\exists y)(0 < y < x)$

5. $(\forall x)(\exists y)(x + y = 10)$

6. $(\exists x)(\forall y)(x \cdot y = y)$

Uniqueness

1. Prove that there is a unique number x such that $x^2 = 0$.

2. Prove that there is a unique number x such that $3x + 5 = 23$.

3. Prove that the solution to $x^2 = 4$ is not unique.

4. Prove that there is a unique solution to $x^2 = 2x + 1$.

Maxima and Minima

1. How can you say "There is no largest integer"?

2. How can you say "There is no smallest positive real number"?

   Let $S \subset \mathbb{Z}$ and $T \subset \mathbb{Z}$ be finite sets, so that $\max(S), \min(S), \max(T)$, and $\min(T)$ are all well-defined.

3. Prove: $\max(S \cup T) \geq \max(S)$. (Let $s = \max(S)$, then show that $s \leq \max(S \cup T)$).

4. Prove: $\max(S \cap T) \leq \max(S)$.

5. Let $-S = [-s | s \in S]$. Prove: $\min(-S) = -\max(S)$. (Start with "let s = max(S)...", then show that $-s = \min(-S)$.)

6. (Harder) Prove that $\max(S \cup T) = \max(\max(S), \max(T))$.

Games

1. Let M = {rock, paper, scissors}, and let $D(x, y)$ stand for "x defeats y." How can you say "No move in rock-paper-scissors is guaranteed to win" in quantifier notation?

2. Now let N = M ∪ {tiger claw}, where tiger claw never loses. How can you say "There is a move that never loses" in quantifier notation?