Chapters 1.4-1.5: Quantifiers

Monday, Juney 29

English to Quantifier

1. There is a unique number x such that $x^2 = 0$. $(\exists! x)(x^2 = 0)$, or $(\exists x)(x^2 = 0 \wedge (\forall y)(y^2 = 0 \rightarrow y = x))$.
2. 1 is the smallest positive integer. $(\forall x \in \mathbb{Z}^+)(1 \leq x)$
3. Every integer is either odd or even. $(\forall x \in \mathbb{Z})(x \in E \vee x \in O)$, where E = even integers, O = odd integers.
4. Either all integers or odd, or all integers are even. $(\forall x \in \mathbb{Z})(x \in O) \vee (\forall x \in \mathbb{Z})(x \in E)$. Note that two separate for-all statements are needed here.
5. Any number divisible by 4 is also divisible by 2. Let F = numbers divisible by 4, E = even numbers. $(\forall x \in F)(x \in E)$.
6. The sum of any two even numbers is even. $(\forall x \in E)(\forall y \in E)(x + y \in E)$.
7. For any $n \geq 2$ there is a prime number between n and $2n$. $(\forall n \geq 2)(\exists p \text{ prime})(n < p < 2n)$.
8. Every even number greater than 2 can be written as the sum of two primes. $(\forall n > 2 \text{ even})(\exists p, q \text{ prime})(p + q = n)$
9. Everybody doesn't like something, but nobody doesn't like Sara Lee. (2 propositions)
   (a) $(\forall x)(\exists y)$(x does not like y)
   (b) $\neg(\exists x)$(x does not like Sara Lee), or
   (c) $(\forall x)$(x likes Sara Lee)
10. Everybody loves my baby, but my baby don't love nobody but me. (2 propositions, maybe 3)
    (a) $(\forall x)$(L(x,B)), where B = my baby, L(x,y) = "x loves y".
    (b) $(\forall x)(x \neq M \rightarrow \neg L(B, x))$, where M = me
    (c) L(B,M)
11. The previous lyrics are from an old song popularized by Louis Armstrong. Prove: if we take "everybody" literally, then "my baby" and "me" must actually be the same person!
    (a) Everybody loves my baby (given)
    (b) Therefore, my baby loves my baby (instantiation)
    (c) My baby does not love anybody but me (given)
    (d) If my baby loves x then x is me (equivalent)
    (e) Since my baby loves x, we conclude that x = me.

Quantifier to English

Write each of the following statements in English and decide whether they are true or false. The domain is $\mathbb{R}$ unless otherwise specified.

1. $(\forall x)(x^2 > 0)$

   For every real number x, x^2 is positive. FALSE: $x = 0$ is a counterexample.

2. $(\exists x)(\forall y)(x > y)$

   There is some x that is greater than all real numbers y. FALSE: x is never greater than itself, so we instead conclude $(\forall x)(\exists y)(x \leq y)$.

3. $(\forall x)(\exists y)(x > y)$

   For every x there is some y such that x is greater than y. TRUE: for any x, let $y = x - 1$.

4. $(\forall x > 0)(\exists y)(0 < y < x)$

   For every positive number x there is a smaller positive number y. TRUE: for any x, let $y = x/2$.

5. $(\forall x)(\exists y)(x + y = 10)$

   For any x there is a y such that $x + y = 10$. TRUE: let $y = 10 - x$.

6. $(\exists x)(\forall y)(x \cdot y = y)$

   There is some x such that $x \cdot y = y$ for every y. TRUE: $x = 1$ is the unique answer.

Uniqueness

1. Prove that there is a unique number x such that $x^2 = 0$.

   First, $0^2 = 0$, so such a number x exists.

   Then if $y^2 = 0$ we know that $y = 0$ or $y = 0$, so $y = 0$ is the only possibility. Therefore the solution is unique.

2. Prove that there is a unique number x such that $3x + 5 = 23$.

   If $3x + 5 = 23$ then $3x = 18$ and so $x = 6$, meaning that there is at most one solution.

   Then plugging in $x = 6$ shows that $3 \cdot 6 + 5 = 18 + 5 = 23$, so 6 is a solution.

   Alternately, we could say that if $y \neq 6$ then $3y \neq 18$ and so $3y + 5 \neq 23$.

3. Prove that the solution to $x^2 = 4$ is not unique.

   2 and -2 are both solutions.

4. Prove that there is a unique solution to $x^2 = 2x - 1$. (TYPO CORRECTED: originally said $x^2 = 2x + 1$)

   If $x^2 = 2x - 1$, then $x^2 - 2x + 1 = 0$, so $(x - 1)^2 = 0$ and $x = 1$. This means that there is at most one solution.

   Then since $1^2 = 2 \cdot 1 - 1$, 1 is a valid solution.

Maxima and Minima

1. How can you say "There is no largest integer"?

   For every $x \in \mathbb{Z}$ there is a $y \in \mathbb{Z}$ such that $y > x$. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x)$.

2. How can you say "There is no smallest positive real number"?

   For every positive real number there is a smaller real number.

   $(\forall x > 0)(\exists y > 0)(y < x)$.

   Or: $(\nexists x > 0)(\forall y > 0)(y \geq x)$

   Let $S \subset \mathbb{Z}$ and $T \subset \mathbb{Z}$ be finite sets, so that $\max(S), \min(S), \max(T)$, and $\min(T)$ are all well-defined.

3. Prove: $\max(S \cup T) \geq \max(S)$. (Let $s = \max(S)$, then show that $s \leq \max(S \cup T)$).

   Let $s = \max(S)$. Then $s \in S$, so $s \in S \cup T$, so by definition of max, $s \leq \max(S \cup T)$.

4. Prove: $\max(S \cap T) \leq \max(S)$.

   Let $s = \max(S \cap T)$. Then $s \in S \cap T$, so $s \in S$. Therefore $s \leq \max(S)$.

5. Let $-S = [-s | s \in S]$. Prove: $\min(-S) = -\max(S)$. (Start with "let s = max(S)…", then show that $-s = \min(-S)$.)

   Let $s = \max(S)$. Then $s \in S$, so $-s \in -S$ by definition of $-S$.

   Then let $t \in -S$ be an arbitrary element. Then $-t \in S$, so $-t \leq s$ by definition of max. Therefore (multiplying by -1), $-s \leq t$.

6. (Harder) Prove that $\max(S \cup T) = \max(\max(S), \max(T))$.

   Let $x = \max(S \cup T)$. Then either $x \in S$ (in which case $x \leq \max(S)$) or $x \in T$ (in which case $x \leq \max(T)$). In either of these cases, $x \leq \max(\max(S), \max(T))$. Therefore $\max(S \cup T) \leq \max(\max(S), \max(T))$.

   Let $m = \max(\max(S), \max(T))$ for convenience. Either $m = \max(S)$ or $m = \max(T)$, but in either case $m \in S \cup T$, and so $m \leq \max(S \cup T)$. Therefore $\max(S \cup T) \geq \max(\max(S), \max(T))$.

   Since $m \geq x$ and $m \leq x$, we conclude that the two are equal.

Games

1. Let M = {rock, paper, scissors}, and let $D(x, y)$ stand for "x defeats y." How can you say "No move in rock-paper-scissors is guaranteed to win" in quantifier notation?

   $(\forall x \in M)(\exists y \in M)(\neg D(x, y))$.

2. Now let N = M ∪ {tiger claw}, where tiger claw never loses. How can you say "There is a move that never loses" in quantifier notation?

   $(\exists x \in M)(\forall y \in M)(\neg D(y, x))$.