Chapters 2.1-2.2: Sets

Friday, June 26

Set Building and Operations

1. Let the universe U be the integers. A be the set of all prime numbers. Let B be the set of even numbers. Let C be the set of all numbers greater than 10. Find an element in each of the following:

   - $A \cap B$: 2 (in fact, $A \cap B = \{2\}$, the singleton set)
   - $A \cap C$: 11, 13, 17, 19, 23,...
   - $B \cap C$: 12, 14, 16...
   - $A \cap \overline{C}$: 2, 3, 5, 7

2. Continuing with the above problem, how many elements are in A? $A \cap B$? $A \cap B \cap C$?

3. If U is the universal set, what are $\overline{U}$ and $\overline{\emptyset}$?

$$\overline{U} = \emptyset, \overline{\emptyset} = U$$

4. Find the elements of the set $[a, b] - (a, b)$.

$$[a, b] - (a, b) = \{a, b\}$$

5. Let $A = \{2z + 1 | z \in \mathbb{Z}\}$. Let $B = \{3z | z \in \mathbb{Z}\}$. Describe the set $A \cap B$ in words.

   $A \cap B$ is the set of all odd numbers divisible by 3.

6. Express the following sets using $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and set builder notation:

   (a) The set of all even positive numbers. $[2z | z \in \mathbb{Z}, z > 0]$

   (b) The set of all non-zero real numbers. $\mathbb{R} - \{0\}$

   (c) $\{1, 2, 4, 8, 16, \ldots\} = [2^n | n \in \mathbb{N}]$

   (d) The set of all real numbers that are not integers. $\mathbb{R} - \mathbb{Z}$

   (e) The set of all ordered pairs of integers whose sum is 10. $[(x, y) \in \mathbb{Z} \times \mathbb{Z} | x + y = 10]$

7. Let $U = \{1, 2, 3, 4\}$ (the universe), $A = \{2, 3\}$, $B = \{1, 2\}$. Express each of the following sets in terms of A, B and set operations.

   (a) $\{2\} = A \cap B$

   (b) $\{1, 2, 3\} = A \cup B$

   (c) $\{1, 4\} = \overline{A}$

   (d) $\{4\} = U - (A \cup B) = U - A - B$

   (e) $\{1, 2, 4\} = U - (A - B)$

8. Let A, B and C be three generic sets. Draw Venn diagrams for each of the following:

   (a) $A \cap (B - C)$

   (b) $A \cap (\overline{B} \cup \overline{C})$

   (c) $A - (B \cup C)$

   (d) $A \cup \overline{B} - \overline{C}$

9. What can you say about the sets A and B if
   (a) $A \cup B = A$? $B \subset A$
   (b) $A - B = A$? $B \cap A = \emptyset$
   (c) $A - B = B - A$? $A = B$
   (d) $A \cap B = A$? $A \subset B$
   (e) $A \cup B = B \cup A$? Nothing at all–this is true for any two sets.

Relation to Propositional Logic

10. What rule in propositional logic does the identity $A \cap U = A$ correspond to? What does U correspond to?

    $A \cap U = A$ is equivalent to $a \wedge \mathbf{T} \equiv a$. The universal set U corresponds to $\mathbf{T}$.

11. What does $\emptyset$ correspond to?

    $\emptyset$ corresponds to $\mathbf{F}$. One example: $A \cup \emptyset = A$ corresponds to $a \vee \mathbf{F} \equiv a$

Proofs

Prove the following:

12. $(-\infty, a] \subset (-\infty, b]$ if and only if $a \leq b$.

    Suppose $a \leq b$. Then if $x \in (-\infty, a]$, $x \leq a$ by definition and so $x \leq b$, meaning $x \in (-\infty, b]$. Thus $x \in (-\infty, a] \Rightarrow x \in (-\infty, b]$ and so $(-\infty, a] \subset (-\infty, b]$.

    On the other hand, suppose $a > b$. Then $a \in (-\infty, a]$ but $a \notin (-\infty, b]$.

13. $(-\infty, a] \subset (-\infty, b)$ if and only if $a < b$.

    Let $A = (-\infty, a], B = (-\infty, b)$.

    Suppose $a < b$. Then if $x \in A$, $x \leq a < b$ and so $x < b$, meaning that $x \in B$. Therefore $A \subset B$.

    If $a \geq b$ then $a \in A$ but $a \notin B$.

14. $A \subset B$ if and only if $A \cap B = A$.

    Suppose $A \subset B$. It is clear that $A \cap B \subset A$ (if $x \in A$ and $x \in B$ then $x \in A$), so we just need to show that $A \subset A \cap B$. If $x \in A$, then $x \in B$, so x is in both A and B, meaning $x \in A \cap B$. This shows that $A \subset A \cap B$.

    On the other hand, suppose that $A \not\subset B$. This means that there exists $x \in A$ such that $x \notin B$. But this means that $x \notin A \cap B$, which shows that $A \cap B \neq A$.

    Putting the forward and inverse statements together, we conclude that $A \subset B$ if and only if $A \cap B = A$.

15. $A \subset B$ if and only if $A \cup B = B$.

    Suppose $A \subset B$. It is clear that $B \subset A \cup B$ (if $x \in B$ then $x \in A$ or $x \in B$), so we just need to prove that $A \cup B \subset B$. If $x \in A \cup B$, then either $x \in B$ (in which case we are done) or $x \in A$, which implies that $x \in B$ (since $A \subset B$). Therefore if $x \in A \cup B$, it follows that $x \in B$, and so $A \cup B \subset B$.

    If $A \not\subset B$, then there exists $x \in A$ such that $x \notin B$. But then $x \in A \cup B$, so $A \cup B \neq B$.

    Putting the forward and inverse statements together, we conclude that $A \subset B$ if and only if $A \cup B = B$.

There is a second way to prove this: $A \subset B$ is equivalent to $\overline{B} \subset \overline{A}$ and $A \cup B = B$ is equivalent to $\overline{A} \cap \overline{B} = \overline{B}$. Then renaming $\overline{B} = C$, $\overline{A} = D$, we get that we need to prove that $D \subset C$ if and only if $D \cap C = C$, which is the same claim as in the previous problem. Therefore, if we have proved one of these two problems, we have proved the other.

16. If $A \subset B$ and $B \subset C$ then $A \subset C$. What rule of inference does this correspond to?

    Suppose $x \in A$. Then $x \in B$ (since $A \subset B$) and so $x \in C$ (since $B \subset C$). Therefore $A \subset C$.

    This corresponds to the rule Hypothetical Syllogism (if $p \to q$ and $q \to r$ then $p \to r$).

17. If $A \subset B$ and $\overline{A} \subset C$ then $B \cup C = U$. What rule of inference does this correspond to?

    Take some $x \in U$. We consider two cases: either $x \in A$ or $x \notin A$ (in which case $x \in \overline{A}$).

    If $x \in A$ then $x \in B$ and so $x \in B \cup C$.

    If $x \in \overline{A}$ then $x \in C$ and so $x \in B \cup C$.

    Either way, $x \in B \cup C$ for any $x \in U$ and so $B \cup C = U$. This corresponds to (a variation of) the rule of Resolution: if $p \to q$ and $\neg p \to r$ then $q \lor r$ is always true.