Chapters 1.7-1.8: Proofs

Thursday, June 25

Proof by Cases

Prove each of the following:

1. 101 is prime (you only need to check for divisors up to 11...why?)
2. For any integer n, $n^3 - n$ is divisible by 2.
3. For any integer n, $n^3 - n$ is divisible by 3.
4. For any integer n, either n^2 is divisible by 4 or $n^2 - 1$ is divisible by 4.
5. There are no integers a and b such that $a^2 + b^2 = 23$.
6. $|a|^2 = a^2$
7. $\max(a, b) \geq \min(a, b)$.
8. $\max(a, b) + \min(a, b) = a + b$
9. $\max(a, b) - \min(a, b) = |a - b|$.
10. $\max(a, b) = \dfrac{a + b + |a - b|}{2}$.
11. (Harder) $|a + b| \leq |a| + |b|$ for all a and b.

Arrow Diagram

Draw a diagram showing how these statements about a real number x relate to each other:

1. $x = 2$ or $x = -2$
2. $x = 2$
3. $x^2 = 4$
4. $x^2 - 4 = 0$
5. x is even
6. $x > 0$

Biconditionals

State the four forms (forward, inverse, converse, and contrapositive) of each of these statements. Which forms appear to be simplest to prove?

1. $x = 3$ if and only if $x^2 = 9$ and $x \neq -3$.
2. n is even if and only if $3n + 3$ is odd.
3. $x^2 = 1$ if and only if $x = 1$ or $x = -1$ (one direction requires a proof by cases).
4. $\max(a, b) = \min(a, b)$ if and only if $a = b$.
5. $|a| = 0$ if and only if $a = 0$.
6. $x^2 = 0$ if and only if $x = 0$.

Backwards Reasoning

1. Prove that $(x - 3)^2 + (x + 3)^2 = 2(x + 3)(x - 3) + 36$
2. Prove that $(a + b - c)^2 = (a + b)^2 + (a - c)^2 + (b - c)^2 - a^2 - b^2 - c^2$.

It's the harmonic-geometric-arithmetic-quadratic mean inequality! If x and y are non-negative real numbers, prove that

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}},$$

with equality at each step if and only if $x = y$ (Prove the three inequalities one at a time).