Chapters 1.7-1.8: Proofs

Thursday, June 25

Proof by Cases

Prove each of the following:

1. 101 is prime (you only need to check for divisors up to 11. . . why?)

   Technically, this is a proof by exhaustion (or brute force) rather than by caes, but it would go as follows:

   101 is not divisible by 2.

   101 is not divisible by 3.

   101 is not divisible by 4.

   101 is not divisible by 5.

   101 is not divisible by 6.

   101 is not divisible by 7.

   101 is not divisible by 8.

   101 is not divisible by 9.

   101 is not divisible by 10.

2. For any integer n, $n^3 - n$ is divisible by 2.

   Break this one into cases depending on whether n is even or odd: First suppose that $n = 2k$ is even. Then $n^3 - n = 8k^3 - 2k = 2(4k^3 - k)$ is also even.

   If $n = 2k + 1$ is odd, then

$$\begin{aligned} n^3 - n &= (2k+1)^3 - (2k+1) \\ &= 8k^3 + 12k^2 + 6k + 1 - 2k - 1 \\ &= 8k^3 + 12k^2 + 4k \\ &= 2(4k^3 + 6k^2 + 2k), \end{aligned}$$

   and so is again even. This completes the proof.

   A second method would be to use the fact that the sum of two even numbers is even and the sum of two odd numbers is even:

   If n is even then n^3 is even and so $n^3 - n$ is even.

   If n is odd then n^3 is also odd and so $n^3 - n$ is even.

3. For any integer n, $n^3 - n$ is divisible by 3.

   We will make our lives simpler by factoring $n^3 - n$ as $n(n+1)(n-1)$.

   If n is of the form $3k$ then $n(n+1)(n-1) = 3k(3k+1)(3k-1)$, which is divisible by 3.

   If n is of the form $3k+1$ then $n(n+1)(n-1) = (3k+1)(3k+2)(3k) = 3k(3k+1)(3k+2)$, which is divisible by 3.

   If n is of the form $3k+2$ then $n(n+1)(n-1) = (3k+2)(3k+3)(3k+1) = 3(k+1)(3k+2)(3k+1)$, which is divisible by 3.

4. For any integer n, either n^2 is divisible by 4 or $n^2 - 1$ is divisible by 4.

   We could take on the four cases $n = 4k, 4k+1, 4k+2, 4k+3$ separately, but as it turns out we only need to look at whether n is odd or even.

   If $n = 2k$ is even then $n^2 = 4k^2$, which is divisible by 4.
   If $n = 2k+1$ is odd then $n^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k$, which is again divisible by 4.

5. There are no integers a and b such that $a^2 + b^2 = 23$.

   Use the fact that if $|a| \geq 5$ or $|b| \geq 5$ then $a^2+b^2 > 23$, so a and b must be in the set $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. This means that the only possibilities for a^2 and b^2 are $0, 1, 4, 9$, and 16.

   We could check all 25 possible pairs individually, but we don't have to do that much work: if $a^2 \leq 9$ and $b^2 \leq 9$ then $a^2 + b^2 \leq 18$, so we need either a^2 or b^2 to be 16…make it a^2. But then this implies that $16 + b^2 = 23$, meaning $b^2 = 7$. Since this has no integer solutions, there are no integers a and b such that $a^2 + b^2 = 23$.

6. $|a|^2 = a^2$

   If $a \geq 0$ then $|a|^2 = a^2$.

   If $a < 0$ then $|a|^2 = (-a)^2 = a^2$.

7. $\max(a, b) \geq \min(a, b)$.

   If $a \geq b$ then $\max(a, b) = a \geq b = \min(a, b)$.

   If $a < b$ then $\max(a, b) = b > a = \min(a, b)$.

   Either way, $\max(a, b) \geq \min(a, b)$.

8. $\max(a, b) + \min(a, b) = a + b$

   If $a \geq b$ then $\max(a, b) = a$ and $\min(a, b) = b$, so $\max(a, b) + \min(a, b) = a + b$.

   If $a < b$ then $\max(a, b) = b$ and $\min(a, b) = a$, so $\max(a, b) + \min(a, b) = b + a = a + b$.

9. $\max(a, b) - \min(a, b) = |a - b|$.

   Suppose $a \geq b$. Then $\max(a, b) = a$, $\min(a, b) = b$, and $a - b \geq 0$ (so $|a - b| = a - b$). Therefore $\max(a, b) - \min(a, b) = a - b = |a - b|$.

   Suppose $a < b$. Then $\max(a, b) = b$, $\min(a, b) = a$, and $a - b < 0$ (so $|a - b| = b - a$). Therefore $\max(a, b) - \min(a, b) = b - a = |a - b|$.

10. $\max(a, b) = \dfrac{a + b + |a - b|}{2}$.

    If $a \geq b$ then $\max(a, b) = a$ and $|a - b| = a - b$.

    Therefore,

$$
\begin{aligned}
\frac{a + b + |a - b|}{2} &= \frac{a + b + a - b}{2} \\
&= \frac{2a}{2} \\
&= a \\
&= \max(a, b)
\end{aligned}
$$

    If $a < b$ then $\max(a, b) = b$ and $|a - b| = b - a$, so

$$
\begin{aligned}
\frac{a + b + |a - b|}{2} &= \frac{a + b + b - a}{2} \\
&= b \\
&= \max(a, b)
\end{aligned}
$$

11. (Harder) $|a+b| \le |a| + |b|$ for all a and b.

    One method: Since $|a+b| \ge 0$ and $|a| + |b| \ge 0$, we use the rule that (when $x \ge 0$ and $y \ge 0$) $x^2 \le y^2$ if and only if $x \le y$. In this particular case, we can say that

$$\begin{aligned} |a+b| \le |a| + |b| &\Leftrightarrow |a+b|^2 \le (|a|+|b|)^2 \\ &\Leftrightarrow (a+b)^2 \le |a|^2 + 2|a||b| + |b|^2 \\ &\Leftrightarrow a^2 + 2ab + b^2 \le a^2 + 2|a||b| + |b|^2 \\ &\Leftrightarrow ab \le |a||b| \end{aligned}$$

    So our desired conclusion is true if and only if $ab \le |a||b|$ for all real a and b. Now the cases are a little simpler:

    If $a \ge 0$ and $b \ge 0$ then $|a||b| = ab$.

    If $a < 0$ and $b < 0$ then $|a||b| = (-a)(-b) = ab$.

    If exactly one of a and b is negative then $ab < 0 \le |a||b|$.

    In all of these cases, $ab = |a||b|$. By our prior chain of reasoning, this implies that $|a+b| \le |a| + |b|$.

Arrow Diagram

Draw a diagram showing how these statements about a real number x relate to each other:

1. $x = 2$ or $x = -2$
2. $x = 2$
3. $x^2 = 4$
4. $x^2 - 4 = 0$
5. x is even
6. $x > 0$

Statements 1, 3, and 4 are all equivalent. 2 implies 1 (and therefore also 3 or 4). Any of 1,2,3, or 4 implies 5. Only 2 implies 6.

$$(5) \Leftarrow (4) \Leftrightarrow (3) \Leftrightarrow (1) \Leftarrow (2) \Rightarrow (6)$$

Biconditionals

State the four forms (forward, inverse, converse, and contrapositive) of each of these statements. Which forms appear to be simplest to prove?

Remark 1 *There is some ambiguity as to which statement should be the "forward" condition, and it would be a little more clear if we had written $A \Leftrightarrow B$ instead of "A if and only if B." But in any case, it doesn't matter–you can pick any version you like for the forward condition because you will have to prove both directions anyway.*

1. $x = 3$ if and only if $x^2 = 9$ and $x \neq -3$.

   Forward: If $x = 3$ then $x^2 = 9$ and $x \neq -3$.

   Inverse: If $x \neq 3$ then $x^2 \neq 9$ or $x = -3$.

   Converse: If $x^2 = 9$ and $x \neq -3$, then $x = 3$.

   Contrapositive: If $x^2 \neq 9$ or $x = -3$, then $x \neq 3$.

   The forward and converse are easiest to prove. The forward should be very simple. For the converse, if $x^2 = 9$ then $x = \pm 3$. Since $x \neq -3$, it follows that $x = 3$.

2. n is even if and only if $3n + 3$ is odd. (assume n is an integer)

   Forward: If n is even then $3n + 3$ is odd.

   Inverse: If n is odd then $3n + 3$ is even.

   Converse: If $3n + 3$ is odd then n is even.

   Contrapositive: If $3n + 3$ is even then n is odd.

   The forward and inverse are easiest to prove. If $n = 2k$ is even then $3n + 3 = 6k + 3 = 2(3k + 1) + 1$ is odd. If $n = 2k + 1$ is odd then $3n + 3 = 6k + 6 = 2(3k + 3)$ is even.

3. $x^2 = 1$ if and only if $x = 1$ or $x = -1$ (one direction requires a proof by cases).

   Forward: If $x^2 = 1$ then $x = 1$ or $x = -1$.

   Inverse: If $x^2 \neq 1$ then $x \neq 1$ and $x \neq -1$.

   Converse: If $x = 1$ or $x = -1$ then $x^2 = 1$.

   Contrapositive: If $x \neq 1$ and $x \neq -1$ then $x^2 \neq 1$.

   The forward and converse are easiest to prove. For the forward, we can say $0 = x^2 - 1 = (x + 1)(x - 1)$ and conclude that $x + 1 = 0$ or $x - 1 = 0$.

   The converse is a small proof by cases: If $x = 1$ then $x^2 = 1$. If $x = -1$ then $x^2 = 1$. Therefore if $x = \pm 1$ then $x^2 = 1$.

4. $\max(a, b) = \min(a, b)$ if and only if $a = b$.

   Forward: If $\max(a, b) = \min(a, b)$ then $a = b$.

   Inverse: If $\max(a, b) \neq \min(a, b)$ then $a \neq b$.

   Converse: If $a = b$ then $\max(a, b) = \min(a, b)$.

   Contrapositive: If $a \neq b$ then $\max(a, b) \neq \min(a, b)$.

   The converse is easiest: if $a = b$ then $\max(a, b) = a = b = \min(a, b)$.

   Let's try the contrapositive, along with a proof by cases: $If a \neq b$ then $a > b$ or $a < b$. If $a > b$ then $\max(a, b) = a > b = \min(a, b)$. If $a < b$ then $\max(a, b) = b > a = \min(a, b)$.

Remark 2 *This proof would be a good time to say "without loss of generality, $a > b$." There is nothing particularly special about the fact that one is called a and the other b, and the proofs for $a > b$ and $a < b$ are exactly the same, but with the variable names reversed. If we note this, then we only have to prove one of the cases.*

5. $|a| = 0$ if and only if $a = 0$.

   Forward: If $|a| = 0$ then $a = 0$.
   Inverse: If $|a| \neq 0$ then $a \neq 0$.
   Converse: If $a = 0$ then $|a| = 0$.
   Contrapositive: If $a \neq 0$ then $|a| \neq 0$.

   The converse is again easiest to prove: If $a = 0$ then $|a| = a = 0$.

   Let's go with the contrapositive again: If $a \neq 0$ then $a > 0$ or $a < 0$. If $a > 0$ then $|a| = a > 0$. If $a < 0$ then $|a| = -a$, but since $a < 0$ it follows that $-a > 0$, and so $|a| > 0$. Either way, $|a| \neq 0$.

6. $x^2 = 0$ if and only if $x = 0$.

   Forward: If $x^2 = 0$ then $x = 0$.
   Inverse: If $x^2 \neq 0$ then $x \neq 0$.
   Converse: If $x = 0$ then $x^2 = 0$.
   Contrapositive: If $x \neq 0$ then $x^2 \neq 0$.

   For the forward: If $x^2 = 0$ then $x = 0$ or $x = 0$, and therefore $x = 0$.

   Conversely: if $x = 0$ then $x^2 = 0 \cdot 0 = 0$.

Backwards Reasoning

1. Prove that $(x-3)^2 + (x+3)^2 = 2(x+3)(x-3) + 36$.

$$\begin{aligned}(x-3)^2 + (x+3)^2 = 2(x+3)(x-3) + 36 &\Leftrightarrow x^2 - 6x + 9 + x^2 + 6x + 9 = 2(x^2 - 9) + 36 \\ &\Leftrightarrow 2x^2 + 18 = 2x^2 - 18 + 36 \\ &\Leftrightarrow 2x^2 + 18 = 2x^2 + 18\end{aligned}$$

   The last statement is true, so (because our statements were connect by "if and only iff" conditions) we can conclude that the initial proposition is true.

   We could also rewrite this proof as one long string of equalities. This has the advantage of being easier to read, but it would be much harder to prove the equalities in this order:

$$\begin{aligned}(x-3)^2 + (x+3)^2 &= 2x^2 + 18 \\ &= 2x^2 - 18 + 36 \\ &= 2(x^2 - 9) + 36 \\ &= 2(x+3)(x-3) + 36\end{aligned}$$

2. Prove that $(a+b-c)^2 = (a+b)^2 + (a-c)^2 + (b-c)^2 - a^2 - b^2 - c^2$.

$$\begin{aligned}
& (a+b-c)^2 = (a+b)^2 + (a-c)^2 + (b-c)^2 - a^2 - b^2 - c^2 \\
\Leftrightarrow\ & a^2+b^2+c^2+2ab-2ac-2bc = a^2+2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2-a^2-b^2-c^2 \\
\Leftrightarrow\ & a^2+b^2+c^2+2ab-2ac-2bc = a^2+b^2+c^2+2ab-2ac-2bc
\end{aligned}$$

It's the harmonic-geometric-arithmetic-quadratic mean inequality! If x and y are non-negative real numbers, prove that

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}},$$

with equality at each step if and only if $x = y$ (Prove the three inequalities one at a time).
We'll go through the middle inequality here–the rest are up to you. Since $x \geq 0$ and $y \geq 0$, we will use the fact that $a > b$ if and only if $a^2 > b^2$ (when $a, b \geq 0$):

$$\begin{aligned}
\sqrt{xy} &\leq \frac{x+y}{2} \\
\Leftrightarrow xy &\leq \frac{x^2+2xy+y^2}{4} \\
\Leftrightarrow 4xy &\leq x^2+2xy+y^2 \\
\Leftrightarrow 0 &\leq x^2-2xy+y^2 \\
\Leftrightarrow 0 &\leq (x-y)^2
\end{aligned}$$

The last statement is always true. Since the propositions are connected by biconditionals, we conclude that if $x, y \geq 0$ then $\sqrt{xy} \leq \frac{x+y}{2}$.